NOTES

NOTE ON THE SEPARATION AND SOLUTION OF
DIFFUSION TYPE EQUATIONS

SOLUTION of a partial differential equation, such as the diffusion or heat flow equation, is often attempted by separating the variables. One assumes a solution of the equation in the form of a product of a function of position and a function of time. The equation for flow of water in unsaturated soils can be put into the same mathematical form as the diffusion equation and the heat flow equation. When the soil-water diffusivity, \( D \), is constant the flow equation in one dimension can be separated by assuming that \( \theta (x,t) = X(x) T(t) \) where \( \theta \) is the water content, \( X(x) \) is independent of the time, \( t \), and \( T(t) \) is independent of the position, \( x \). The flow equation can then be separated into two ordinary differential equations.

\[
\frac{1}{T} \frac{dT}{dt} = \frac{D}{X} \frac{d^2X}{dx^2} = -\alpha^2 = \text{const.} \quad [1]
\]

The equations are solved separately to give \( X(x) \) and \( T(t) \), which are then multiplied to give \( \theta (x,t) \). The constant \( \alpha^2 \) is evaluated by fitting the solution of the equation in \( x \) to the boundary conditions. For certain types of boundary conditions \( \alpha^2 \) may take on an infinite number of discrete values. Thus, there is a separate solution of the equation for each value of \( \alpha^2 \). It is often possible to match a sum of all these solutions to the initial condition of the problem by proper selection of a coefficient for each term. The complete solution of the problem is then an infinite series of terms which are products of a function of \( x \) and a function of \( t \).

When \( D \) is not constant, which is usually the case in unsaturated soils, difficulty arises. The one-dimensional flow equation is then:

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left( D \frac{\partial \theta}{\partial x} \right) \quad [2]
\]

\( D \) is now a function of \( \theta \) and, therefore, a function of \( x \) and \( t \). In order to separate equation [2], it is necessary that \( D \) as well as \( \theta \) be separable into a function of \( x \) multiplied by a function \( t \). In general, this is not possible. In many flow problems, it is found that after a relatively short time only one term in the infinite series is important and all the others are negligible. In effect, the initial conditions of the problem are damped out and only the present condition and the boundary conditions are important. For certain types of relationships between \( D \) and \( \theta \), \( D \) can be separated if \( \theta \) is separable, e.g., \( D = a \theta^r \). Denoting \( D(x,t) \) by \( D_x(x) D_t(t) \) and \( \theta (x,t) \) by \( X(x) T(t) \), equation [2] becomes

\[
\frac{1}{D_t T} \frac{dT}{dt} + \frac{1}{X} \frac{d}{dx} \left( D \frac{dX}{dx} \right) = -\alpha^2 \quad [3]
\]

We again have two ordinary differential equations and can proceed as above. The solution thus obtained will obey the flow equation and by proper evaluation of \( \alpha^2 \), will match the boundary conditions. Inasmuch as the solution will not necessarily match the initial conditions, it is only approximate. For \( Dt/L^2 > 0.3 \), it should prove to be a reasonable approximation for certain types of boundary conditions.

One process for which the solution of the flow equation results in the product of a function of \( x \) and a function of \( t \) is the outflow from a pressure membrane apparatus. Exact solution of this problem by numerical procedures\(^2\) shows that, except in the vicinity of the plate, the water content is fairly uniform. Over much of the sample we may then assume \( D_x \) to be constant which gives us \( \alpha^2 = D_x \pi^2/4L^2 \) where \( L \) is the length of the sample.\(^3\) From equation [3], we can then write:

\[
\frac{1}{D_t T} \frac{dT}{dt} = \frac{D_x \pi^2}{4L^2} \quad [4]
\]

If we multiply numerator and denominator by \( X(x) \) and rearrange, we get

\[
\frac{1}{D(\theta - \theta_f)} \frac{d\theta}{dt} = -\frac{\pi^2}{4L^2} \quad [5]
\]

where \( D \) is the diffusivity which corresponds to the particular value of \( \theta_f \), and where \( \theta_f \) is the final equilibrium water content. The total rate of outflow is obtained by integrating over \( x \). The contribution to the outflow is least where \( D \) is smallest in the vicinity of the membrane, and it should not be too far in error to replace \( \theta \) by the average water content over the entire sample and use for \( D \) the diffusivity corresponding to that water content. Analysis of the exact solution shows that this is justified when the diffusivity is an exponential function of the water content. If \( D \) is known as a function of \( \theta \), equation [5] can be integrated to give a more exact solution.

A useful application of equation [5] is in the determination of \( D \). If \( W \) is the total water content and \( W_f \) is the final equilibrium water content, then, from equation [5]

\[
D = -\frac{(W - W_f) \pi^2}{4L^2 (dW/dt)} \quad [6]
\]

The diffusivity can be calculated directly from the instantaneous outflow rate, the water content, and the dimensions of the sample. The equation can be applied to outflow from a membrane or to drying by evaporation, as long as no flow is allowed at one boundary and the water content is maintained constant at the other boundary.

The approximation is not restricted to one-dimensional flow and may readily be extended to radial and other geometries. Correction for membrane impedance by successive approximations is possible, although it may be of doubtful utility.—W. R. Gardner, U. S. Salinity Laboratory, Riverside, Calif.
