Reply to “Comments on ‘Fractal Dimensions of Soil Aggregate-Size Distributions Calculated by Number and Mass’”

We thank Dr. A.B. McBratney for his comments on our article (Perfect et al., 1992). In this reply we will clarify the type of fractal dimension used, and provide a physical explanation for observed values > 3.

Fractals are geometrical constructions that exhibit scaling symmetry. The fractal dimension, \( D \), characterizes the scaling properties of such constructions, and is defined as (Mandelbrot, 1982)

\[
D = -\log(N)/\log(1/r)
\]  

where \( N \) is the number of elements in the generator, and \( 1/r \) is the scaling ratio between successive generations. The operation of applying the generator causes either accretion (irregularity) or reduction (fragmentation) of the initiator. The Koch curve and Cantor bar are well-known examples, respectively. Thus, \( D \) measures both irregularity and fragmentation in combination. In the case of fractal reduction of an Euclidean initiator, \( D \) becomes “solely a measure of fragmentation” (Mandelbrot, 1982).

The size distribution of fragments generated by fractal reduction can be characterized by the following relationship (Mandelbrot, 1982; Turcotte, 1986):

\[
N(X \geq x_*) = kx_*^D
\]  

where \( x_* \) is the fragment length, \( N(X \geq x_*) \) is the cumulative number of fragments with lengths \( \geq x_* \) and \( k \) is the number of initiators. In the limit \( x_* \to 0 \), the value of \( D \) in Eq. [2] is identical to that obtained using Eq. [1]. In practice, Eq. [2] is fitted across a finite range of \( x_* \) on a log-log scale, with \( D \) equal to the slope (Turcotte, 1986; Rieu and Sposito, 1992). The value of \( D \) obtained by this method represents an approximation of the theoretical fractal dimension.

Turcotte (1986) used Eq. [2] to derive a fractal reduction model for cubes in which mass is conserved and \( D \) depends on the probability of failure under an applied stress \( \sigma \), \( \{P\}_\sigma \). Assuming \( \{P\}_\sigma \) is scale invariant, it can be shown that

\[
D \approx \log(8\{P\}_\sigma)/\log(2)
\]  

where the allowed range for the fractal dimension is \( 0 \leq D \leq 3 \).

A problem in applying Eq. [3] to fragmented materials such as soils is that observed values of \( D \) can exceed 3 (Turcotte, 1986). In our study, values of \( D \) for the aggregate number-size distributions (obtained by fitting Eq. [2] to the data on a log-log scale using the method of least squares) ranged from 0.67 to 3.92 (Perfect et al., 1992). The mean standard error associated with these fits was 0.15. Altogether, 13% of the distributions gave values of \( D \) significantly greater than 3.

We have proposed (Perfect et al., 1993) a multifractal model for dry aggregate fragmentation in which the probability of failure under an applied stress is given by a hyperbolic function of scale:

\[
\{P\}_\sigma = 1 - q x_*^{-r}; \quad q^{ir} \leq 1
\]

\[
\{P\}_\sigma = 0; \quad 0 \leq x_* \leq q^{i-1} r
\]

where \( 1 - q \) is the probability of failure for the initiator, \( r \) determines the extent of scale dependence, both \( q \) and \( r \) are positive. For an Euclidean initiator, where \( D = 3 \), Eq. [3] reduces to

\[
D \approx \log(8\{P\}_\sigma)/\log(2)
\]

Therefore, \( D \) becomes independent of \( \{P\}_\sigma \) for an Euclidean initiator. Rieu and Sposito (1991) have provided a start in this direction. However, further research is required on fractal and multifractal reduction of fractal initiators. Rieu and Sposito (1991) have shown that \( D \) changed with scale as predicted by Eq. [5] (Perfect et al., 1993b).

We have shown that fractal dimensions determined from Eq. [2] conform with the definition proposed by Mandelbrot (1982). We have described a model for dry aggregate fragmentation in which the probability of failure under an applied stress \( \sigma \), \( \{P\}_\sigma \), is scale invariant, thus, their Eq. [43] is incorrect. For aggregates that fail according to Eq. [4], it can be shown that \( D \) becomes independent of \( \{P\}_\sigma \) for an Euclidean initiator.

Finally, we note McBratney’s concern with our use of Eq. [2] to characterize dry aggregate-size distributions instead of the lognormal function. Rieu and Sposito (1991) discussed the relationship between the fractal number-size distribution and the lognormal mass-size distribution. We have proposed (Perfect et al., 1993a) that the breaking process associated with the lognormal function assumes \( \{P\}_\sigma \) is scale invariant. Thus, it offers a practical advantage over Eq. [2]. Furthermore, Eq. [6b] shows the lognormal function is inconsistent with our data on aggregate-size distributions.

Received 12 Mar. 1993

Dep. of Land Resource Science
Univ. of Guelph,
Guelph, Ontario N1G 2W1
Canada