Filtering of Period Infiltration in a Layered Vadose Zone: 1. Approximation of Damping and Time Lags

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Infiltration and downward percolation of water in the vadose zone are important processes that can define the availability of water resources. We present an approach that provides insight into how periodic infiltration forcings at the land surface filter in a layered vadose zone in terms of changes in the timing and magnitude of hydrologic responses. To represent geologically realistic systems, we used vertical sequences of one-dimensional periodic solutions, where each solution represents a single soil in a layered profile. The overall approach is based on a linearized Richards equation and assumes that the effects on flow of continuous pressure head changes at soil interfaces are negligible. We evaluated the limit of these approximations by comparison with results from the numerical model HYDRUS-1D, which uses the full Richards equation. We compared (i) the depth at which flux variations became steady, and (ii) the travel time of wetting fronts to reach a depth of 3 m. The solution was reasonably accurate (error less than a factor of 2) for infiltration cycles with periods from 30 to 365 d and for fluxes common in arid and semiarid environments (0–2 mm d⁻¹). Lag times between a surface forcing and response at any depth were accurate (error less than a factor of 1.1). The approximation generally provided consistent estimates of the damping and time lag, such that it overestimated the depths where fluxes were steady and underestimated the time for a forcing to reach a specific depth.

Vadose zone flow and recharge to aquifers are some of the most poorly defined fluxes in hydrologic models at local and regional scales (Lerner et al., 1990; Scanlon et al., 2002; Healy, 2010). These processes can be uncertain because of nonlinear relations between flow and the hydraulic properties of soils, which are inherently heterogeneous (Yeh and Harvey, 1990; Warrick and Nielsen, 1980). In addition to the inevitable impacts of soil heterogeneity on vadose flux patterns and uncertainty, there can be additional complexities and uncertainties of flow because pressure head gradients and water contents can change abruptly at boundaries between soils under variably saturated conditions (Zaslavsky, 1964; Yeh, 1989).

Current numerical approaches for assessing the impact of climatic variability on water resources at the watershed (tens to hundreds of square kilometers) and continental scales (millions of square kilometers) can represent complex vadose zone responses to hydroclimatic surface forcings based on atmospheric observations. Examples of models of physically based integrated surface and subsurface processes include SWAT/MODFLOW (Sophocleous and Perkins, 2000), Mike-SHE (Graham and Butts, 2005), HydroGeoSphere (Therrien et al., 2006), Parflow (Maxwell and Miller, 2005; Kollet and Maxwell, 2006), GSFLOW (Markstrom et al., 2015), and CATHY (Niu et al., 2014). However, efforts to represent all heterogeneities, nonlinear soil-water relations among heterogeneous soils, and hydroclimatic forcings in numerical models can be computationally expensive (Kollet et al., 2010) and require large datasets to identify model parameters. Kinematic wave approximations of the Richards equation (Smith, 1983; Niswonger et al., 2006; Morway et al., 2012) simplify soil-water relations to efficiently resolve vadose zone flow from surfacing forcing, but the existing tools are currently limited to homogenous soils. Other approaches use conceptual hydrologic models that can fit many land surface observations (e.g., Liang et al., 1994; Markstrom et al., 2015) but, by simplifying physical processes, can be limited
for investigating the role of the vadose zone for transmitting atmospheric forcings.

An alternative approach for estimating the impacts of surface forcings on vadose zone flow through heterogeneous soils is the use of algebraic solutions to the linearized Richards equation (Philip, 1968; Pullan, 1990). An algebraic solution can be advantageous because (i) predictions can be made without requiring solution for all preceding times, (ii) the solution can be used to represent fewer processes and identify parameters at local scales using smaller datasets rather than using regional models and large datasets, and (iii) a solution is obtained without long computational time (Haitjema, 1995) or spin-up time to obtain initial conditions (Ajami et al., 2014). Some solutions based on the linearized Richards equation, however, might not resolve key flow dynamics and processes because of limitations to steady-state flow (Philip, 1968; Pullan, 1990) or the transient boundary forcings may be restricted to few functional forms. For example, solutions are available for steady flow in homogeneous (Philip, 1968; Raats, 1970) and heterogeneous soils (Warrick and Yeh, 1990; Yeh et al., 1985; Yeh, 1989; Rockhold et al., 1997; Warrick and Knight, 2003). Solutions for transient flow are mainly available for homogeneous soil (Warrick, 1975; Lomen and Warrick, 1978), while for heterogeneous soil, boundary forcings may vary as a step change (Srivastava and Yeh, 1991). Heterogeneity and boundary conditions can also be represented stochastically (Lu et al., 2007). Despite these limitations, solutions that adequately match system boundaries and processes can provide insight into predictions of interest, which may be difficult to obtain with more computationally expensive numerical approaches. An evaluation of the limitations in algebraic solutions can be used to identify the systems where the solutions are reasonable.

We have developed an approximation for the filtering of periodic forcings in a layered vadose zone. The filtering results in time-lagged responses of water content and flux after a forcing, and damped variability with depth (Bakker and Nieber, 2009; Dickinson et al., 2014; Corona et al., 2018). The solutions are based on the periodic flow solution presented by Bakker and Nieber (2009) for a single homogeneous soil where the flux varies sinusoidally at the land surface and is filtered with depth. This approach assumes that infiltration and percolation in the vadose zone are predominantly vertical and can be represented by the sum of solutions for a steady, long-term component and a variable, periodic component (Townley, 1995; Trefry, 1999; Dickinson et al., 2004; Bakker and Niefber, 2009; Dickinson et al., 2014). We extend their solution for layered soil by including sinusoidal boundaries at the top and bottom of a one-dimensional profile through the layers, and use vertical sequences of the solutions to represent each layer. In addition to the approximations adopted by Bakker and Nieber (2009), we also make the following approximations: (i) the diffusive properties are constant and uniform within each layer, (ii) the filtering is based on responses in overlying soil, and is independent of responses in lower soil, and (iii) the steady component does not vary within a layer. In reality, diffusive properties often change with depth at boundaries between soils because the pressure head is continuous and transitions by changing above a boundary in response to the sharp change in soil properties across the boundary (Zaslavsky, 1964; Bear, 1972). Furthermore, all state variables and state-dependent parameters vary with time in response to transient fluxes, while the functions defining hydraulic conductivity and water content are fixed. With the goal of providing useful insights into the impacts of layers on the filtering of surface forcings on groundwater recharge, we investigated the flux and soil conditions under which the effects of the linearization on the lag time and damping of flux variations are acceptable. We developed appropriate caveats regarding the limits of these interpretations.

Here we present an approximate solution for periodic flow based on the approach of Bakker and Nieber (2009) and Trefry (1999) through vertical sequences of soil layers. We provide examples of the filtering of periodic variations in soils where pressure head and water content transition at soil layer interfaces. Then we explain the simplifications underlying the solution. Finally, we identify conditions of soil type and fluxes where these approximations are acceptable or violated. We base this evaluation on a comparison of the effects of layer boundaries on the periodic component of percolation predicted by the solution compared with predictions made with numerical models that fully account for the effects of soil-water transitions at layer interfaces. Our evaluations are based on relatively simple numerical models that could represent vertical percolation at local scales within a regional system. The solution could be applied in a similar manner to quickly characterize the filtering properties across larger systems where infiltration and soil properties are spatially heterogeneous. Dickinson and Ferre (2018) presents practical applications of the solution for understanding the filtering in regional systems and for model simplification.

Solution for Periodic Flow in Soil

We use vertical sequences of solutions for vertical, sinusoidal flux in homogeneous soil according to Bakker and Nieber (2009) to represent the filtering of periodic flow through soil layers. Bakker and Nieber (2009) provided full details of the solutions. We summarize their solution to review all pertinent approximations and describe the extension to layered soils.

Consider vertical flow in the vadose zone where the z axis is downward positive. Flow is governed by the one-dimensional Richards equation, which can be written as

\[
C \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z} \left( K \frac{\partial \psi}{\partial z} \right) \frac{\partial K}{\partial z}
\]

[1]

where \( \theta \) (dimensionless) is the water content, \( \psi \) [L] is pressure head, \( K(\psi) \) [L T\(^{-1}\)] is the hydraulic conductivity, \( t \) [T] is time, and \( z \) [L] is depth. The water capacity \( C \) [L\(^{-1}\)] is defined as

\[
C = \frac{\partial \theta}{\partial \psi}
\]

[2]
The relation between hydraulic conductivity and pressure head is approximated by the Gardner model (Gardner, 1958):

$$K = K_s \exp \left( \alpha (\psi - \psi_e) \right) \quad \psi < \psi_e$$

where $K_s$ [T−1] is the hydraulic conductivity at saturation, $\alpha$ [L−1] is a fitting parameter based on the pore-size distribution, and $\psi_e$ [L] is the air-entry pressure head. The relation between water content and pressure head is approximated by the Gardner–Kozeny model (Mathias and Butler, 2006):

$$\theta = n_0 \exp \left[ \mu (\psi - \psi_e) \right] \quad \psi < \psi_e$$

where $n_0$ (dimensionless) is the porosity, and $\mu$ [L−1] is a fitting parameter.

A solution is sought for one-dimensional unsaturated flow at depths $z$ and times $t$ where the vertical flux $q_p(z,t)$ [T−1] is the sum of a steady component $q_s$ [T−1] and a periodic component $q_{pr}$ [L−1] with a real part $q_{pr}$ and an imaginary part $q_{pi}$ that represent the amplitude and phase of the periodic flow with time:

$$q = q_s + q_{pr} \exp(i \omega t)$$

where $\omega = 2 \pi / P$ (rad cycle−1) is the angular frequency of the variation, $i$ is the imaginary number, and $P$ [T] is the period of the variation. The periodic component Eq. [5] can be written as (Townley, 1995)

$$q = q_s + q_{pa} \cos(\omega t + \phi)$$

with amplitude of the flux variation $q_{pa} = \sqrt{(q_{pr}^2 + q_{pi}^2)}$ and phase angle $\phi = \tan^{-1}(q_{pi}/q_{pr})$. The change in amplitude with depth, called the damping factor $\delta_z$ (dimensionless) (Townley, 1995; Bakker and Nieber, 2009), is defined as the ratio of the amplitude of the flux variation at depth $z$ to the amplitude of the flux variation $q_{pa}$ at the top of the layer at depth $z = j$:

$$\delta_j = \frac{q_{pa}}{q_{pa,j}}$$

The phase lag $\Delta \phi$ is the difference in the phase at depth $z$ to the phase at depth $z = j$:

$$\Delta \phi = \arg(q_p) - \arg(q_{pa,j})$$

Bakker and Nieber (2009, see their Eq. [10] and [20]) linearized the Richards equation using the Kirchoff potential $H$ [L2 T−1] and obtained solutions for steady and periodic components of $H(z,t)$:

$$H = H_s + H_p \exp(i \omega t)$$

The steady component of $H$ within the $j$th soil layer is approximated to be a constant:

$$H_{sj} = \frac{K_j}{\alpha_j} = \frac{q_s}{\alpha_j}$$

The periodic component $H_p = H_{pr} + iH_{pi}$ obtained by the approach of Townley (1995) and Bakker and Nieber (2009) is

$$H_p(z) = A_1 \exp \left[ \alpha_j (z - z_j) + i \omega t \right]$$

$$+ A_2 \exp \left[ \alpha_j (z - z_{j+1}) + i \omega t \right]$$

where $z_j$ is the depth of the top of the $j$th layer, $z_{j+1}$ is the depth to the bottom of the $j$th layer, and $A_{1j}$ and $A_{2j}$ are integration constants for the $j$th layer to be determined for specific physical boundaries.

Equation [11] has two solutions:

$$a_{1j} = \frac{1}{2} \left( \frac{1}{2} \alpha_j - \frac{4i \omega D_j}{D_j} \right)^{1/2}$$

and

$$a_{2j} = \frac{1}{2} \left( \frac{1}{2} \alpha_j - \frac{4i \omega D_j}{D_j} \right)^{1/2}$$

where $a_{1j}$ and $a_{2j}$ are complex numbers with real parts $a_{1rj}$ and $a_{2rj}$ and imaginary parts $a_{1ij}$ and $a_{2ij}$. The diffusivity $D_j$ [L2 T−1] can be written as

$$D_j = \frac{K_j}{\mu n_0}$$

where $\mu n_0$ is the water content at steady flow, $q_s$:

$$\theta_{st} = n_0 \left( \frac{q_s}{K_s} \right)^{\mu/\alpha}$$

The real parts of $a_{1j}$ and $a_{2j}$ relate to the damping of the amplitude of the flux variation with depth, and the imaginary parts relate to the phase lag of the flux variation with depth. The term $a_{1j}$ is positive and increases the damping and phase lag upward from the bottom of a layer. That is, the damping and phase lag of flux variations can propagate toward the center of a layer from the top and bottom boundaries.

Following Bakker and Nieber (2009), a solution for $H$ in a single $j$th soil layer can be obtained by substituting Eq. [10] into Eq. [9]:

$$H = \frac{q_{s}}{\alpha_j} + A_1 \exp \left[ \alpha_j (z - z_j) + i \omega t \right]$$

$$+ A_2 \exp \left[ \alpha_j (z - z_{j+1}) + i \omega t \right]$$

A specified flux boundary condition indicates

$$q = - \frac{\partial H}{\partial z} + \alpha H = nq$$

where $n$ equals 1 at the top of a layer (positive flux in) and −1 (negative flux out) at the bottom of a layer. Equations [16] and [17] can be combined to provide a solution for $q$ with steady and periodic components for the $j$th layer:

$$q = q_s + \alpha_j A_{1j} \exp \left[ \alpha_j (z - z_j) + i \omega t \right]$$

$$+ \alpha_j A_{2j} \exp \left[ \alpha_j (z - z_{j+1}) + i \omega t \right]$$
Bakker and Nieber (2009) defined a solution domain for a single soil layer between \( z = 0 \) at the land surface and an infinite depth \( (z = \infty) \) where the steady flux is \( q_s \) and the periodic boundary conditions are

\[
q(z_j = 0, t) = q_{pj+1} \exp(i \omega t) \quad [19]
\]

\[
q(z_{j+1} = \infty, t) = 0 \quad [20]
\]

That is, the amplitude of the periodic flux at the land surface is \( q_{pj+1} \) and the amplitude is completely damped at infinite depth. They determined that these boundary conditions required solutions only for negative values for \( a \) and the integration constant \( A_j \). For a single layer with boundary conditions defined by Eq. [19] and [20] (Bakker and Nieber, 2009), the solutions for the integration constants are

\[
A_1 = q_{pj+1} / (1 - a_1) \quad \text{and} \quad A_2 = 0.
\]

A soil with periodic boundaries at the top and bottom uses solutions for \( q(z, t) \) with \( a_1 \) and \( a_2 \) and integration constants \( A_j \) and \( A_2 \) to define Eq. [16] and [18] for each layer. We use Eq. [19] to define \( q_{pj+1} \), the periodic boundary condition for the top of the \( j \)th layer at depth \( z_j \) and the boundary condition at the bottom of the layer at depth \( z_{j+1} \) is

\[
q(z_{j+1}, t) = q_{pj+1} \exp(i \omega t) \quad [21]
\]

Solutions for \( A_1 \) and \( A_2 \) can be obtained by setting Eq. [19] and [21] both equal to the periodic part of Eq. [18]:

\[
A_1 = \frac{q_{pj} - q_{pj+1} \exp(-a_2 \alpha_x L)}{\alpha_j (1 - a_1) \left| 1 - \exp\left(a_1 \alpha_x L - a_2 \alpha_x L\right) \right|} \quad [22]
\]

\[
A_2 = \frac{q_{pj+1} - q_{pj} \exp(a_1 \alpha_x L)}{\alpha_j (1 - a_2) \left| 1 - \exp(a_1 \alpha_x L - a_2 \alpha_x L) \right|} \quad [23]
\]

where the layer thickness \( L_j (z - z_j) \).

The solution for \( H \) is obtained by substituting \( A_1 \) and \( A_2 \) into Eq. [16] and taking the real part:

\[
H = \frac{q_s}{\alpha_j} + \Re\left( \frac{q_{pj} - q_{pj+1} \exp(-a_2 \alpha_x L)}{\alpha_j (1 - a_1) \left| 1 - \exp\left(a_1 \alpha_x L - a_2 \alpha_x L\right) \right|} \times \exp\left[a_1 \alpha_x (z - z_j) + i \omega t\right] \right)
\]

\[
+ \left[\frac{q_{pj+1} - q_{pj} \exp(a_1 \alpha_x L)}{\alpha_j (1 - a_2) \left| 1 - \exp(a_1 \alpha_x L - a_2 \alpha_x L) \right|} \times \exp\left[a_2 \alpha_x (z - z_{j+1}) + i \omega t\right]\right) \quad [24]
\]

The solution for \( q \) is found by substituting \( A_1 \) and \( A_2 \) into Eq. [18] and taking the real part:

\[
q = q_s + \Re\left( \frac{q_{pj} - q_{pj+1} \exp(-a_2 \alpha_x L)}{\alpha_j (1 - a_1) \left| 1 - \exp\left(a_1 \alpha_x L - a_2 \alpha_x L\right) \right|} \times \exp\left[a_1 \alpha_x (z - z_j) + i \omega t\right] \right)
\]

\[
+ \left[\frac{q_{pj+1} - q_{pj} \exp(a_1 \alpha_x L)}{\alpha_j (1 - a_2) \left| 1 - \exp(a_1 \alpha_x L - a_2 \alpha_x L) \right|} \times \exp\left[a_2 \alpha_x (z - z_{j+1}) + i \omega t\right]\right) \quad [25]
\]

**Approximation for Layered Soil**

Steady and periodic flow in soil layers is represented by a vertical series of solutions using Eq. [24] and [25]. We follow the approach of Trefry (1999) to determine the integration constants by matching the flux boundary conditions at the interfaces of layers. Trefry (1999) extended the approach of Townley (1995) by using a series of solutions for confined flow to represent one-dimensional periodic flow through adjacent sub-aquifers. We use Trefry's notation when possible for convenience.

Consider a vertical sequence of \( N \) soil layers \( j \), which are internally homogeneous (Fig. 1). Each soil layer is bound by interfaces with layers \( j - 1 \) and \( j + 1 \) or by a boundary at the top of layer \( j = 1 \) or bottom of layer \( j = N \). The depths of the \( N - 1 \) interfaces between layers \( j \) and \( j + 1 \) are \( z_{j+1} \), and the solution domain for each layer is between depths of \( z - z_j \) and \( z - z_{j+1} \).

A consequence of Eq. [10] is that \( H_j \), the steady component of the Kirchoff potential \( H \) in each layer, is approximated to be spatially constant. We used this approximation to solve an ordinary differential equation for a transient, periodic flow solution to obtain Eq. [11] for each layer using the approach of Bakker and Nieber (2009, Eq. [10] and [20]). The approximation of a constant \( H_j \) in each layer allows diffusivity in Eq. [12] and [13] to also be constant within each layer. This requires the pressure head to be treated as discontinuous between layers. Thus, the periodic flow solution does not consider transitions in pressure head to achieve continuity at layer boundaries or the changes in diffusivity that result from transitions in pressure head. This approximation results in simplifications of Eq. [24] and [25] in this application to layered soils. Bakker and Nieber (2004a, 2004b) solved for spatially variable \( H \) around circular inhomogeneities, but for steady flow.

The solution for flow through all soil layers is provided by \( N \) sets of equations for the steady components \( q_s \) and \( H_j \), and \( N \) sets of Eq. [16] and [18] for the periodic component. The solutions for \( q \) and \( H \) are a linear sum of the steady and periodic components, so solutions for the steady and periodic components can be identified separately. The steady components can be determined from flux continuity and \( \alpha_j \) in each \( j \)th layer. The solutions for \( N \) sets of periodic solutions require values for \( A_1 \) and \( A_2 \), which can be obtained by the matrix approach described by Trefry (1999).
where the damping and phase shift over overlying layers 1 to 2

\[ \delta_j = \exp \left( a_1 \alpha_j \left( z - z_j \right) \right) = \exp \left( - \frac{(z - z_j)}{\lambda_j} \right) \]  

where the characteristic length \( \lambda \) is \( \text{(Bakker and Nieber, 2009)} \)

\[ \lambda_j = \frac{1}{a_1 \alpha_j} \]

The damping in overlying layers \( l = 1 \) to \( l = j - 1 \) with thickness \( L_j \) can be accumulated into a single term \( \gamma_j \):

\[ \gamma_j = \sum_{l=1}^{j-1} a_{1l} \alpha_l L_l \]  

The phase lag \( \Delta \varphi_j \) at depth \( z \) within the \( j \)th layer can be written as (Dickinson et al., 2014)

\[ \Delta \varphi_j = a_{1j} \alpha_j \left( z - z_j \right) \]

\[ = -\alpha_j \left( z - z_j \right) \]

\[ \times \left[ 1 + \frac{4 \omega}{\alpha_j^2 D_j} \right]^{1/4} \sin \left( \frac{1}{2} \arctan \left( \frac{4 \omega}{\alpha_j^2 D_j} \right) \right) \]  

The phase lag from overlying layers \( l = 1 \) to \( l = j - 1 \) is accumulated into a single term \( \phi_j \):

\[ \phi_j = \sum_{l=1}^{j-1} a_{1l} \alpha_l L_l \]  

The phase lag at depth \( z \) in the \( j \)th layer is determined from the accumulated phase lag from overlying layers and the phase lag in the \( j \)th layer from the top of the layer, \( z_j \), to depth \( z \):

\[ \Delta \varphi_j = \phi_j + a_{1j} \alpha_j \left( z - z_j \right) \]

where \( a_{1j} \) is the imaginary part of \( a_j \) for the \( j \)th layer. The damping factor and phase lag at the top of the \( j \)th layer, where \( z = z_j = 0 \), are equal to \( \exp(\gamma_j) \) and \( \exp(i\phi_j) \), respectively. For the top layer \( (j = 1) \), \( \gamma_{j=1} = 0 \) and \( \phi_{j=1} = 0 \).
The value of \( H_{p,(j+1)} \) at the top of layer \( j + 1 \) can be approximated from Eq. [16] using the solutions for \( A_1 \) and \( A_2 \) for the boundary conditions applied by Bakker and Nieber (2009):

\[
H_{p,(j+1)} = \frac{q_{p,j}}{\alpha_j (1 - a_j)} \exp(\gamma_j) \exp(i\phi_j) \tag{35}
\]

where \( q_{p,j} \) is the periodic component at depth \( z = 0 \). The value of \( H_{p,j+1} \) at the bottom of the \( j \)th layer is approximated as

\[
H_{p,j+1} = \frac{q_{p,j}}{\alpha_j (1 - a_j)} \exp(\gamma_j + a_{L_j} \alpha_j L_j) \times \exp[i(\phi_j + a_{L_j} \alpha_j L_j)] \tag{36}
\]

The exponential terms in Eq. [35] and [36] are equal, so \( H_{b,j\text{diff}} \) at the interface of layers \( j \) and \( j + 1 \) is

\[
H_{b,j\text{diff}} = q_{p,j} \frac{1}{\alpha_j (1 - a_j)} \frac{1}{\alpha_{j+1} (1 - a_{j+1})} \times \exp(\gamma_j \gamma_j + a_{L_j} \alpha_j L_j) \exp[i(\phi_j + a_{L_j} \alpha_j L_j)] \tag{37}
\]

Following Trefry (1999), the matrix approach includes equations for the boundary conditions and for layer interfaces. The boundary conditions can be represented by equations of the form

\[ p_A + \varepsilon A_2 = \rho. \]

A periodic flux boundary at the top of layer \( j = 1 \), where \( z = 0 \), can be written as \( \beta = \alpha_1 (1 - a_1) \), \( \varepsilon = \alpha_1 (1 - a_{21}) \), and \( \rho = q_{p,1} \). A periodic \( H \) boundary at the bottom of layer \( j = 1 \) (\( z = L_1 \)) could be written as \( \beta = \exp(a_{L_j} \alpha_j L_j) \), \( \varepsilon = \exp(a_{2j} \alpha_j L_j) \), and \( \rho = H_{p,j+1} \), where \( H_{p,j+1} \) is the specified periodic \( H \) boundary at depth \( L_1 \).

Equations for continuity of flux at interfaces from Eq. [21] and [25] can be written as

\[
\alpha_j a_{L_j} (1 - a_{L_j}) \exp(a_{L_j} \alpha_j L_j) + \alpha_{L_j} a_2 \exp(a_2 \alpha_j L_j) - \alpha_{j+1} a_{L_{j+1}} (1 - a_{L_{j+1}}) - \alpha_{L_{j+1}} a_2 (1 - a_{2j+1}) = 0 \tag{38}
\]

The difference in the Kirchoff potential \( H_p \) at interfaces can be written using Eq. [11] and [27]:

\[
A_j \exp(a_{L_j} \alpha_j L_j) + A_{L_j} \exp(a_{2j} \alpha_j L_j) - A_{j+1} a_{L_{j+1}} = H_{b,j\text{diff}} \tag{39}
\]

The matrix equation can be written as (Trefry, 1999):

\[
MA = V \tag{40}
\]

where \( A^T = (\alpha_{11}, \alpha_{21}, ..., \alpha_{1j}, \alpha_{2j}, ..., \alpha_{1N}, \alpha_{2N}) \), are the integration constants, \( V^1 = (\rho_1, ..., H_{b,j\text{diff}} 0, ..., \rho_N) \), and

\[
M = \begin{bmatrix}
\beta_1 & \varepsilon_1 & 0 \\
\vdots & \ddots & \vdots \\
\exp(a_{L_j} \alpha_j L_j) & \exp(a_{2j} \alpha_j L_j) & -1 & -1 \\
\alpha_j (1 - a_j) \exp(a_{L_j} \alpha_j L_j) & \alpha_j (1 - a_{2j}) \exp(a_{2j} \alpha_j L_j) & -\alpha_{j+1} (1 - a_{L_{j+1}}) & -\alpha_{L_{j+1}} (1 - a_{2j+1}) \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \beta_N & \varepsilon_N
\end{bmatrix}
\tag{41}
\]
The 365 d to represent annual wet and dry seasons (Fig. A). In both examples, the flux varies symmetrically around 10⁻⁴ m d⁻¹ (1.83 mm yr⁻¹) and 1.95 flux in a homogeneous silty clay. The other example involves the soil layers switched. The 3.62 0.096 0.48 7.3 4 2.18 −0.33
Sandy loam 2.67 0.38 0.39 11.21 3.33 −0.18

Table 1. Gardner (G) (Gardner, 1958) and van Genuchten (vG) (van Genuchten, 1980) parameters α, saturated hydraulic conductivity (Kₛ), saturated water content (θᵣ) at porosity nₛ, fitting parameter µ, and air-entry pressure head (ψₑ) for silty clay and sandy loam in the examples.

atmospheric or hydrologic data (e.g., Hanson et al., 2004; Gurdak et al., 2007; Dickinson et al., 2014; Velasco et al., 2017). The fine soil is a silty clay and the coarse soil is a sandy loam, which have differing soil hydrologic parameters (Table 1) that produce complex flow and water content responses at the layer boundary. The soil hydraulic properties of the soils were defined by the Gardner (1958) and Gardner–Kozeny (Mathias and Butler, 2006) soil models. We translated van Genuchten parameters for the soils described in the Rosetta soil catalog (Schaap et al., 2001) to Gardner soil parameters by following the procedure described by Wraith and Or (1998) and implemented by Dickinson et al. (2014).

Filtering of Variations from the Periodic Solution

We demonstrate the filtering of cyclical infiltration in layered soils by representing the flow in each layer by the solution. One example has an upper layer of silty clay and lower layer of sandy loam (Fig. B). The other example has the soil layers switched. The steady component of the flux, qₛ, is held constant at 1 × 10⁻⁴ m d⁻¹ and the amplitude, qₚₑ, is equal to 0.95qₛ. That is, the surface flux ranges between 0.05 × 10⁻⁴ m d⁻¹ (1.83 mm yr⁻¹) and 1.95 × 10⁻⁴ m d⁻¹ (71.2 mm yr⁻¹). The flux varies with a period P of 365 d to represent annual wet and dry seasons (Fig. 3A). In both examples, the depth to the soil interface (bottom of the upper soil) is 1.66 m, which is approximately half the damping depth for this flux in a homogeneous silty clay.

Profiles of the flux with depth at times P/4, P/2, 3P/4, and P illustrate how the flux varies within each soil and with depth (Fig. C). In both examples, the flux varies symmetrically around the mean flux. The flux variability damp more with depth in the silty clay than the sandy loam, and the rate of damping with depth changes at the layer boundary. When the upper soil is a relatively fine silty clay (Fig. 3B), about 22% of the surface forcing is preserved at the layer boundary at depth 1.66 m, and the flux variations damp to 5% of the surface forcing in the underlying sandy loam at 8.59 m, which is the damping depth. Thus, most of the damping occurs in the silty clay. When the upper soil is sandy loam (Fig. 3C), a larger amount of surface forcing, 65%, remains at the layer boundary. Then, flux variations damp over a shallower depth range in the lower fine soil, and the damping depth is 4.8 m. In both of these cases, most of the damping occurs in the fine soil regardless of the soil type at the surface. That is, surface soil information may be insufficient for predicting how surface forcings filter with depth and affect recharge.

In the solution results, the rate of the damping is constant with depth in each layer because ψ and C transitions at the layer boundary are not represented. Thus, the solution does not include the effects of layering on soil-water properties between the layers, and damping is determined using only the thickness and soil hydraulic parameters of the overlying soil layers. For example, the amount of damping is the same in both Fig. 3B and 3C at depth 3.32 m (dashed horizontal line), twice the thickness of the upper layer. At this depth, we sum the damping across equal depth ranges using linearized solutions for damping in both soils, and the order of the soil sequence does not affect the damping. That is, the solution computes the filtering of surface forcings for any soils at any depth by considering each soil independently, which may simplify investigations of the filtering properties of real systems. We evaluate whether the transitions at layers and the soil order affects the filtering below.

Cyclical variations in infiltration at the land surface (z = 0) produce wetting fronts that propagate downward through time. Wetting fronts from prior infiltration cycles are peaks in the flux at lower depths. When the upper soil is silty clay (Fig. 3B), the uppermost wetting front at time P/2 remained in the silty clay and propagated to a depth of 1 m, while the next lower peak moved to depth 4.5 m in the sandy loam. When the upper soil is sandy loam (Fig. 3C), the first wetting front propagated deeper, near the soil transition at 1.6 m, but the second peak in the silty clay moved less far (4 m). In either case, the wetting front propagated deeper while in the sandy loam, indicating that the wave speed is greater

Fig. 2. The procedure for computing the damping depth d in layered sandy loam and silty clay soils. The heavy black line is the damping factor δ at depth z in each jth layer. The long-dashed line is the damping factor that would extend below layer j = 1 if the underlying layers did not exist. The short-dashed line is the damping factor that would extend below layer j = 2 if the bottom layer j = 3 did not exist. The potential damping depth for each soil occurs when the damping factor equals 0.05. Because the potential damping depths for j = 1 and j = 2 are below the layer bottoms, neither are the damping depth. The damping depth for j = 3 is within layer j = 3, so it is the actual damping depth.
in the sandy loam, identified as a slight steepening and elongation of the flux profiles in the sandy loam. The second peak traveled more through the lower layer in both cases, so the wave speed of the lower layer controls the total lag time. Again, information for only the surface soil may be insufficient to understanding the timing between a surface forcing and recharge.

**Effect of Pressure-Head Transition on Damping**

We explored how the transition in \( \psi \) and the soil-water properties at the layer interface can affect the damping depth using results from the numerical model. We used an example with two soil layers where \( \psi \) transitions widely between the layers. The steady component of the flux, \( q_s \), is held constant at \( 0.5 \times 10^{-3} \) m d\(^{-1} \), and the amplitude of the variation, \( q_{pa} \), is equal to \( 0.45 \times 10^{-3} \) m d\(^{-1} \). That is, the surface flux ranges between \( 0.025 \times 10^{-3} \) m d\(^{-1} \) (9.125 mm yr\(^{-1} \)) and \( 0.975 \times 10^{-3} \) m d\(^{-1} \) (355.88 mm yr\(^{-1} \)). The infiltration at the surface varies cyclically with a period \( P \) of 90 d to represent two annual wet and dry seasons. The ranges of the computed variations of flux, \( q_s \), and \( C \) with depth are indicated by shaded envelopes, which are widest (most variable) near the land surface (\( z = 0 \)) and narrow (damp) with increasing depth (Fig. 4).

To examine the specific effects of the transition, the soil properties are selected so the assigned soil properties that affect the damping \( (\alpha, \theta, \text{ and } D \text{ in Eq. [3], [4], and [11])} \) are identical in the same areas above and below the transition. The upper and lower layers are both silty clay with identical Garder and Gardner-Kozeny soil properties (Table 1), but to obtain different \( \psi \), the layers were synthetically assigned different air-entry pressure heads \( \psi_e \). That is, the system is designed to be essentially homogeneous, but the resolved \( \psi \) at the specified flux is different in each layer (Fig. 4B) and the time-averaged soil-water properties in the upper and lower soil layers outside of the \( \psi \) transitions are the same. When \( \psi \) increases with depth in the transition, the time-averaged \( \theta \), \( D \), and \( C \) also increase with depth in the transition. When \( \psi \) decreases in the transition, the time-averaged soil-water properties also decrease. The upper layer has \( \psi_e = -0.33 \) m while the lower layer has \( \psi_e \) equal to 0 or \(-0.66 \) m. Dark gray shading surrounds the variations with depth when \( \psi_e = 0 \) m in the lower layer, and white indicates that \( \psi_e = -0.66 \) m. The thickness of the \( \psi \) transition is \( z_T \), defined here as the length above the layer interface where 1% of the transition remains. At the specified flux, the pressure head transitions from \( \psi = -1.05 \) m in the upper layer to \( \psi = -0.72 \) m across a distance \( z_T = 0.83 \) m when \( \psi_e = 0 \) m in the lower layer. When \( \psi_e = -0.66 \) m in the lower layer, \( \psi \) transitions to \(-1.38 \) m across \( z_T = 0.5 \) m.

For these same soils and surface flux, the solution, which neglects the \( \psi \) transition, predicted a damping depth of 2.55 m. The numerical model, however, indicated that the damping either increased or decreased because of the transition in the soil-water properties across distance \( z_T \) (Fig. 4). The damping was greater across the transition when \( \psi \) increased in the transition (Fig. 4A, dark shaded area). Conversely, when \( \psi \) decreased, the damping was less across the transition, viewed as less narrowing of variations within the transition (Fig. 4A, white area). The damping depth was shallower, 2.11 m, when \( \psi_e = 0 \) m in the lower layer (white) and 2.47 m when \( \psi_e = -0.66 \) m. Dickinson et al. (2014) found that the solution overestimated damping in homogeneous soils. Here, the solution overestimated the damping in both cases, but it was more similar when the pressure head transitioned to a lower value in the underlying layer. That is, the solution may be more accurate in cases where transitions in pressure heads result in deeper damping depths and less filtering of surface forcings.

The transition affects the damping because of changes in the water capacity \( C \), defined as the resistance to change in \( \psi \) per
If $C$ is large, a change in $y$ will result in a relatively large amount of water added to storage, causing the medium to absorb more of the flux variability. This results in greater damping. If $C$ is small, there is more resistance to water going into storage, and the soil absorbs less of the flux variability. This reduces the damping. Thus, $y$ transitions between soils can result in either more damping if $y$ increases with depth in underlying soils or less damping if $y$ decreases in a transition at a layer boundary.

These effects are more pronounced when the pressure head and water capacity difference between the soils are large and in highly heterogeneous systems where the time-averaged pressure head and the water capacity are continuously transitioning within different soils. The pressure head and water capacity do not transition in the approximate solution, thus the damping may be overestimated in systems where the pressure head transitions to higher values in underlying soils, or even underestimate the damping if the pressure head is continuously decreasing with depth in lower layers.

**Using a Numerical Model to Evaluate the Validity of Assumptions in the Approximate Solution**

We evaluated the approximate solution for layered soils and a range of cyclical fluxes that are typical of arid and semiarid environments. Conditions in these environments are challenging for evaluating the solution because water contents and other soil-water properties can be highly variable between long drying periods and episodic infiltration events. We chose soils that have large transitions in soil-water properties at layer boundaries. The infiltration fluxes are similar to seasonal, annual, or irrigation cycles observed in hydrologic time series.

We evaluated the damping depth and variations in flux, as well as the soil-water properties, in silty clay and sandy loam through comparison to numerical results using HYDRUS-1D (Šimůnek et al., 2005). The damping depth from the numerical model, $d_{num}$, is computed as the depth where the variation in the flux from the numerical model is reasonably equal to 5% of the imposed variation at depth $z = 0$ m. The lag time for wetting fronts (waves) to reach a depth of $z = 3$ m from the solution, $\tau_{3m}$, is compared with that of the numerical model, $\tau_{3m, num}$. The travel time for a wetting front to reach depth $z = 3$ m is computed from the numerical results by tracking the downward movement of flux peaks with time from depth $z = 0$ m to depth $z = 3$ m. In these comparisons, $q_s$ ranges from $1 \times 10^{-4}$ to $1 \times 10^{-3}$ m d$^{-1}$ and $P$ ranges from 30 to 365 d.

We assume that the numerical model results are more accurate because they explicitly consider that the diffusivity changes as a nonlinear function of $\theta$ (Dickinson et al., 2014) with depth and time. We define the solution to be acceptably accurate if the error
in the damping depth is within a factor of 2. Dickinson et al. (2014) evaluated whether the approximation of constant diffusivity was reasonable for predictions of the damping depth in homogeneous soil. They used the same threshold for homogeneous soils and found that the solution overestimates the damping depth but that the error is acceptable for fine-textured and some coarse soils when the flux cycles produce small changes in water content and diffusivity with time. In this investigation, we also tested how the transition of soil-water properties at layer boundaries affect the applicability of the solution. Computational run times of the numerical models, including a spin-up period, ranged from several seconds to a minute, whereas the computational time of the solution was less than a second.

**Evaluation of Variations with Depth**

We compare vertical profiles of the flux, pressure head \( \psi \), water content \( \theta \), diffusivity \( D \), and water capacity \( C \) at times \( P/4 \) and \( 3P/4 \) (Fig. 5 and 6) in the upper 10 m of the vadose zone from the approximate solution and numerical models. Figure 5 shows a silty clay that overlies sandy loam, and Fig. 6 shows the soil layers switched. The numerical solution shows the actual response of the layered system. Namely, flux and water pressure were continuous across the boundary. In this case where a silty clay layer overlies sandy loam, the damping was less in the lower layer. Both solutions represent a continuous flux across the boundary. In contrast, \( \psi \) changed abruptly near the boundary and was discontinuous across the two layers. In this case, \( \psi \) must be higher in the upper layer (leading to a higher water content, which resulted in a higher relative hydraulic conductivity) for the lower soil to accept the same flux as the upper soil. The magnitude of this difference depended on the flux and the hydraulic properties of the two materials. The \( \psi \) transitioned within a relatively small distance above the boundary to establish a response that resembled that in

![Fig. 5. Comparison of profiles of](p. 10 of 16)
a homogeneous soil. Beneath the layer boundary, the underlying soil has essentially no zone of transition. The water content, diffusivity, and water capacity responses had similar responses at the layer boundaries. A fundamental assumption in the solution is that damping and time lags are primarily due to effects outside of these transition zones. Consequently, the solution may lead to unacceptably inaccurate predictions if the transition zones exert a significant influence on damping and time lag. Such environments can have thin layers of alternating fine and coarse soils where the pressure head is always transitioning to that of the underlying soil. Other conditions have soils in which large transitions in pressure head (for example, 0.5 m) between layers results in a diffusivity that is poorly represented by the solution.

The flux profiles were nearly identical near the land surface, but the flux variability from the numerical model damped more quickly with depth than the variability from the solution, resulting in a shallower damping depth from the numerical model (6.06 m) than the solution (8.59 m). In a homogeneous soil, Dickinson et al. (2014) found that the water content calculated by the numerical model (Fig. 5C) decreased more during dry times of the flux cycle than it increased during the wet times. That is, the water content skewed toward lower values and the time-averaged water content at any depth was lower than the average water content in the solution. Thus, the time-averaged diffusivity (Eq. [14]), based on the water content at steady flux $q_s$ (Eq. [15]), was lower in the numerical model. Diffusivity was constant (not skewed) in the solution results, so the time-averaged diffusivity was higher than that of the numerical model, and a higher diffusivity in the solution resulted in an overestimated damping depth. The damping depth was reasonably accurate if the water content variation was relatively small and a constant diffusivity with time was realistic (Dickinson et al., 2014). Dickinson et al. (2014) applied the solution of Bakker Fig. 6. Comparison of profiles of (A) flux $q_z$, (B) pressure head $\psi$, (C) water content $\theta$, (D) diffusivity $D$, (E) water capacity $C$, and envelopes that surround the variations produced by the approximate (white envelope) and numerical (dark gray envelope) solutions. The soils are sandy loam overlying silty clay. The flux period is 365 d, and steady flux $q_s = 1 \times 10^{-4}$ m/d. The flux profiles from the solution are the same as those shown in Fig. 3A. The depth of the soil interface is 1.66 m. The pressure head transitions across thickness $z_T = 0.30$ m above the layer boundary. In areas above and below the pressure head transition, the time average of diffusivity is greater than that of the numerical model. Within the transition, the approximate solution overestimates the water capacity. These approximations result in an overpredicted damping depth, but the overprediction is less than shown in Fig. 5 for silty clay over sandy loam.
and Nieber (2009) for a homogeneous soil to identify areas where the damping depth was above or below the depth of the water table in a regional aquifer. If the damping depth was above the depth of the water table, they stated that recharge could be approximated to be steady. Otherwise, recharge could be assumed to be transient. The overprediction of the damping depth in homogeneous soils suggested that the approximate solution is conservative if it is used to identify areas within which recharge can be considered to be constant with time. That is, if the solution indicates that the cyclical components will dampen at a fixed water table depth, the reality is that the fluctuations will damp at an even shallower depth.

Examining the profiles in Fig. 5 more closely, the approximate solution and numerical model differed the most near the soil layer interface at depth 1.66 m. In the numerical model results, the pressure head increased (Fig. 5) and decreased (Fig. 6) smoothly above the bottom of the lower layer (Zaslavsky, 1964; Bear, 1972) as it transitioned between \( \psi \) values of the lower and upper layers across a thickness \( z_T = 0.84 \) m (Fig. 5) and \( z_T = 0.30 \) m (Fig. 6). In the lower layer, \( \psi \) varied around a mean value of \(-0.92\) m (Fig. 5) and \(-1.23\) m (Fig. 6). The transitions of \( \psi \) at the bottom of the upper layer occur with gradual increases (Fig. 5) and decreases (Fig. 6) of \( 0 \) (Eq. [4]), \( D \) (Eq. [14]), and \( C \) (Eq. [2]) as determined by the Gardner (1958) and Gardner–Kozeny (Mathias and Butler, 2006) soil-water relations. Then, \( 0, D, \) and \( C \) abruptly change to the resolved values at \( \psi \) in the lower layer. In the case of silty clay over sandy loam, the solution underestimated \( C \) within the depth of the \( \psi \) transition, which led to less resistance to variations of \( \psi \). This misrepresentation of \( C \) contributed to deeper transmission of the forcings, and the solution overestimated the damping depth. In the case of the sandy loam over silty clay, \( C \) is overestimated at the layer boundary. This contributed to shallower transmission of the forcings, and the damping depths of the solution and numerical models are more similar.

In layered soils, the damping depth from the solution and numerical models differ for both reasons—the overestimation of diffusivity and the misrepresentation of the water capacity at the layer interface by the solution. In Fig. 5, the solution overestimates the damping depth because the water capacity is too low in the transition and because the time-averaged diffusivity is too high. Thus, both mechanisms compound the error in the damping depth. The overestimation of diffusivity may be an acceptable approximation if the water content variations are relatively small (Bakker and Nieber, 2009). In Fig. 6, \( \psi \) decreases within the transition, and the solution overestimates the water capacity in the transition, which reduces the damping depth. Thus, the damping depth error is reduced if both mechanisms produce offsetting errors. In other layered systems that have small differences in \( \psi \) at layer boundaries, the effects of the transition are likely to be small but the solution will generally be conservative and overestimate the damping depth. In cases where \( \psi \) becomes much lower in an underlying layer, the solution could underestimate the damping depth. However, the case we tested here (sandy loam over silty clay) has a relatively large reduction in \( \psi \), and the solution remains a conservative estimator of the damping depth.

### Evaluation of Damping Depths

We identified the soil and flux conditions where the approximate solution is reasonably accurate by comparing its damping depths to those of the numerical solution in layered silty clay and sandy loam soils. We show contours of the relative error using the ratio of the damping depth produced by the solution and numerical models, \( d/d_{\text{num}} \) (Fig. 7). This ratio is the factor by which the solution overestimated the damping depth. For example, a ratio of 2 indicates that it overestimated the damping depth by a factor of 2. In these examples, the steady component of the flux \( q_s \) ranged from \( 1 \times 10^{-4} \) to \( 1 \times 10^{-3} \text{ m d}^{-1} \) and the period \( P \) ranged from 30 to 365 d. In the numerical solution, \( q_{pa} \) is equal to 95% of \( q_s \), while the damping depth computed by the solution was independent of \( q_{pa} \) (Bakker and Nieber, 2009; Dickinson et al., 2014). The damping depths from HYDRUS-1D (Šimůnek et al., 2005) were a numerical approximation and affected by model discretization, thus \( d_{\text{num}} \) has some error and the contours of \( d/d_{\text{num}} \) can be jagged.

In Fig. 7A, the upper soil is silty clay and the lower soil is sandy loam. In Fig. 7B, the soils switch. The depth to the soil interface is fixed at 10 m, thus the damping depth can occur in either the upper or lower layer under different flux conditions. The heavy white line follows the contour where the damping depth is 10 m. The ratios \( d/d_{\text{num}} \) that plot to the lower left side of the white line are the error when the damping depth occurs in the upper layer, while the ratios on the opposite side are the errors of the damping depth in the lower layer.

The error generally increases as the both the steady flux \( q_s \) and \( P \) increase in both soil configurations. That is, the error compounds across a greater depth as the damping depth increases. The error is also shown along sections that cross the plots at \( q_s = 0.55 \text{ m d}^{-1} \) and \( P = 197.5 \text{ d} \). These sections show that the error abruptly changes when the damping depth is near the layer boundary. When silty clay is the upper soil (Fig. 6A), the ratio \( d/d_{\text{num}} \) is relatively small (<1.1) when the damping depth is in the silty clay (area shaded gray). As \( q_s \) and \( P \) both increase, the damping depth moves into the sandy loam and the ratio increases to about 2.2 when \( q_s = 1 \times 10^{-3} \text{ m d}^{-1} \) and \( P = 365 \text{ d} \). When the soils are switched (Fig. 6B), the ratio is higher in the upper sandy loam, >1.2, and increases to 1.5 in the silty clay when \( q_s = 1 \times 10^{-3} \text{ m d}^{-1} \) and \( P = 365 \text{ d} \). For these soil and flux conditions, the error is most often within the defined threshold of 2. That is, the damping depth is not more than twice the depth from the numerical solution. The solution generally overestimates the damping depth in layered soils, indicating that for the fluxes and periods tested here, the solution is a conservative estimator of the damping depth. In other words, it can only identify areas of steady recharge to be transient and cannot identify areas of transient recharge to be steady.

The error for any specific damping depth arises because the solution approximates the diffusivity to be a constant, while
it varies spatially and temporally in the numerical model. Here, the amplitude $q_{pa}$ of the flux forcing is $0.95q_{s}$, so the flux varies across a larger range at larger $q_{s}$. This increased amplitude of flux variations increases the variability of diffusivity and reduces the time-averaged diffusivity in the numerical model that the solution cannot represent. Larger period $P$ also results in a wider range of water content and diffusivity at depth because more time is available for the system to equilibrate to the variable flux forcing at the surface. Thus, the error is generally lower under conditions where the water content and diffusivity are relatively constant (Dickinson et al., 2014). From our analysis, such conditions have a lower amplitude and shorter period (30 d or less) of flux boundary variation.

A closer look at the error near the soil interface shows how the transition of $\psi$ and the soil-water properties can affect the validity of the approximations in the solution. The error can either increase or decrease when the damping depth is just below the layer interface (along the upper right side of the white line). These changes in the error are visible as sharp changes in the error along the sections. In Fig. 6A, the ratio $d/d_{\text{num}}$ increases abruptly from approximately 1.2 at the interface to about 1.8 in the lower sandy loam (white areas in sections in Fig. 6A). In this case, pressure head $\psi$ increases at the transition, but the solution underestimates the water capacity $C$, as also shown in Fig. 5. Consequently, there was less resistance to changes in flux and pressure head, so the forcings persist deeper and the solution overestimated the damping depth. The error decreases as the damping depth extends deeper into the lower layer. That is, the transition becomes less important as small differences in flux near the damping depth are farther away from the layer boundary. For the case of sandy loam over silty clay, (Fig. 6B), the ratio $d/d_{\text{num}}$ decreases sharply from 1.25 for damping depths at the interface to <1.1 just below the interface. Here, $\psi$ decreases and $C$ increases at the interface, leading to a more shallow damping depth from the solution. This results from a lower resistance to changes in $\psi$ in the solution, which reduces the damping depth. For these soils and flux configurations, the damping depth from the solution remains greater than that of the numerical model.

The approximation of soil-water properties at the soil interfaces introduces error that can increase or decrease the damping depth. We tested the solution for a 10-layer soil of alternating silty clay and sandy loam for the same ranges of $q_{s}$ and $P$ in Fig. 7. The errors were nearly the same as for the two-layer examples, suggesting that an overestimate of the damping by one soil is offset by an underestimate from the other soil.

### Evaluation of Lag Time

We used the same soil layers and thicknesses shown in Fig. 5 and 6 and the same fluxes in Fig. 7 to evaluate the lag time computed by the approximate solution (Fig. 8). We evaluated the lag time for a surface forcing to reach depths $z = 3$ m (Fig. 8A and 8B) and $z = 10$ m (Fig. 8C and 8D). The error is indicated as the ratio of the lag times of the solution and numerical models as $\tau/\tau_{\text{num}}$, where $\tau_{\text{num}}$ is the lag time from the numerical solution. Ratios near 1.0 indicate close agreement between the solutions, and ratios >1.0 indicate that the lag time of the analytical solution was greater than that of the numerical solution due to slower wave propagation.
speed. Because the soil interface is at 1.66 m, the errors at depth \( z = 3 \text{ m} \) reflect an infiltration pulse that traveled through nearly equal depths in the silty clay and sandy loam. That is, a large part of the flow path was in the \( \psi \) transition, and the transition had a strong effect on the errors at the 3-m depth. The errors at depth \( z = 10 \text{ m} \) include fewer effects of the transition on the ratio because the infiltration pulse had moved farther from the transition.

The error ratio ranged from 0.9 to 1.1 at depth \( z = 3 \text{ m} \) in all flux and soil cases that we tested (Fig. 8). The ratios were closer to 1.0, between 0.97 and 1.02, at depth \( z = 10 \text{ m} \). That is, the solution both overestimated and underestimated the lag time and the wave speed of the infiltration pulse at both depths. The ratios of the lag times for each case in Fig. 8 generally decreased with increasing \( q_s \) and \( P \). Thus, the lag times of the solution became shorter relative to that of the numerical solution. The differences between the solutions at low \( q_s \) (Fig. 8A and 8C) were lower because the flux variations were mostly damped out at shallower depths above the transition for lower \( q_s \) and \( P \), so the transition had less influence on the lag time.

Overall, the differences between actual lag times were relatively small compared with the forcing period. The solution overestimated the lag time by 35 d for the 365-d forcing at the 3-m depth for an overlying soil of silty clay and underestimated the lag time by 22 d for the 365-d forcing at the same depth for an overlying sandy loam. These differences in lag times were nearly the same at the 10-m depth, indicating that little additional error accumulated as the wetting front percolated with depth.

The lag time errors were generated by the same processes that contribute to the damping depth error—a lower time-averaged diffusivity in the numerical solution that cannot be matched by the solution (Dickinson et al., 2014) and abrupt increases and decreases of diffusivity at \( \psi \) transitions at layer interfaces. To investigate the effects of a lower time-averaged diffusivity, we examined the lag times for a homogenous silty clay, which had no transition. We found that the error ratio of the homogeneous soil also decreased with increasing \( q_s \) and \( P \). At longer \( P \), the flux and water content varied more widely as the flux changed slowly and moved toward equilibration between the extremes of the forcing. Thus, the lag time differences increased because the numerical model had a lower time-averaged diffusivity. For \( q_s \), the amplitude of the forcing increased with \( q_s \) as 0.95, so the flux varied more widely at larger \( q_s \). This produced a more variable and lower time-averaged diffusivity. These errors are likely to be smaller, and the solution is more representative of real systems that have relatively constant diffusivity from smaller forcing amplitudes and shorter periods of forcing.

To examine the contribution of the \( \psi \) transition on errors, we compared the errors at the 3-m depth (closer to the transition zone) and the 10-m depth. The lag time ratios were most different from 1.0 at 3 m because lag times in the numerical model were both longer and shorter near the transition than those in the solution. The ratios were closer to 1.0 at the 10-m depth because the influence of the transition on the lag time was lower. The ratios were most positive (approximate solution overestimated lag time) when the overlying soil was silty clay (Fig. 8A). In that case, the pressure head and diffusivity both increased in the transition zone in the numerical solution results (Fig. 5) but did not increase in the solution. Increases in diffusivity result in increased wave speed (see Eq. [23], [28], and [42]), so the numerical model produced shorter lag times to reach a depth of 3 m than did the solution. When the upper soil was sandy loam (Fig. 8B), pressure head and diffusivity decreased in the transition (Fig. 6). As a result, the numerical solution produced a slower wave speed and longer lag time relative to the solution. That is, it overpredicted the travel time when pressure heads increased in transition zones and underpredicted travel times when pressure heads decreased in transitions. We found that these effects were largest near the layer interface. At greater distance from the layer boundaries, the transitions become less important, and the solution generally underestimates the lag time. Thus, the actual lag times in a real system would be longer.
The approximation could be included in integrated hydrologic models to obtain initial states of the vadose zone, such as water content and flux, to reduce model spin-up time. The solution provides insight into the processes that lead to time-varying recharge to aquifers and whether the variations are sufficiently damped in the vadose zone and recharge is steady or must be treated as transient.

Overall, the approach is most accurate (ratio $d/d_{\text{num}}$ between 0.8 and 1.2 and $\tau/\tau_{\text{num}}$ between 0.95 and 1.05) for systems where the diffusivity is relatively constant (small changes in water content) and when the pressure head transitions between layers are relatively small. A user can define their own level of acceptability of the solution for estimating damping and time lag. However, the solution can be used to identify areas within which recharge is relatively constant with time or which time lag between infiltration and recharge should be considered for regional-scale hydrologic models. That is, it is a conservative estimator of the damping because it overestimates the damping depth and cannot misidentify areas with transient recharge as having steady-state recharge. This may be a desirable bias because the solution does not incorrectly predict that an area of transient recharge has steady recharge, which could result in inaccurate estimates of recharge variability. If the solution indicates that recharge is transient, the examples demonstrate that the solution can both overestimate and underestimate the lag time. However, the time-lag errors (30 d for a 1-yr cycle and 3 d for a 30-d cycle) were small relative to the period of the variation. Thus, the solution provides reasonable ranges of lag times relative to the period of surface forcings and can help guide the development of surface and unsaturated flow processes in hydrologic models.

**Conclusions**

We developed an approximate approach for estimating the impacts of surface forcings and cyclical climate on vadose zone flow through heterogeneous soils. Our solution is an alternative to computationally expensive models for understanding and conceptualizing how cyclical changes from climate are transmitted to aquifers and if the changes could impact groundwater recharge. The solution provides predictions of time-varying flux and water content at any depth and time without solving for preceding times. The approximation could be included in integrated hydrologic models to obtain initial states of the vadose zone, such as water content and flux, to reduce model spin-up time. The solution provides insight into the processes that lead to time-varying recharge to aquifers and whether the variations are sufficiently damped in the vadose zone and recharge is steady or must be treated as transient.

Our approach uses vertical sequences of the algebraic solutions of Bakker and Nieber (2009) to approximate the filtering of surface infiltration forcings with depth in layered soils. Filtering occurs as a damping of flux variability with depth and a lag time between a forcing and response at any depth. Separate solutions represent the damping in each layer. The approximate solution is based on a linearized Richards equation that approximates diffusivity to be a constant in each layer (Bakker and Nieber, 2009). Thus, transitions of $\psi$, water content, diffusivity, and water capacity between soil layers (Zaslavsky, 1964; Bear, 1972; Yeh, 1989) are not represented. Consequently, the solution misrepresents these soil water properties, which affects its predictions of damping and lag time. We compared the damping depth and lag times of the solution with the results of a numerical model (HYDRUS-1D, Šimůnek et al., 2005), which uses the full Richards equation. We assumed that the numerical model represented real systems where the diffusivity varied at all depths and transitions abruptly between soil layers. Thus, we used these comparisons to identify cases where the approach is likely to be acceptable despite the approximations of constant diffusivity within each layer.

Overall, the approach is most accurate (ratio $d/d_{\text{num}}$ between 0.8 and 1.2 and $\tau/\tau_{\text{num}}$ between 0.95 and 1.05) for systems where the diffusivity is relatively constant (small changes in water content) and when the pressure head transitions between layers are relatively small. A user can define their own level of acceptability of the solution for estimating damping and time lag. However, the solution can be used to identify areas within which recharge is relatively constant with time or which time lag between infiltration and recharge should be considered for regional-scale hydrologic models. That is, it is a conservative estimator of the damping because it overestimates the damping depth and cannot misidentify areas with transient recharge as having steady-state recharge. This may be a desirable bias because the solution does not incorrectly predict that an area of transient recharge has steady recharge, which could result in inaccurate estimates of recharge variability. If the solution indicates that recharge is transient, the examples demonstrate that the solution can both overestimate and underestimate the lag time. However, the time-lag errors (30 d for a 1-yr cycle and 3 d for a 30-d cycle) were small relative to the period of the variation. Thus, the solution provides reasonable ranges of lag times relative to the period of surface forcings and can help guide the development of surface and unsaturated flow processes in hydrologic models.

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**References**


Dickinson, J.E. 2018. Code for use within MATLAB for computing the damping and lag time of responses to cyclical infiltration in a layered vadose zone in Central Valley, California. [Data release.] USGS, Reston, VA. doi:10.5066/P9BH74M


