Bounds to Air-Flow Patterns during Cyclic Air Injection into Partially Saturated Soils Inferred from Extremum States

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Soil-aeration procedures are widely used as an environmental practice for soil and groundwater remediation (removal of volatile compounds and enhancement of biological treatment) and less frequently as an agricultural practice to prevent root O₂ deficiency. Analytical solutions to problems of air flow in the soil allow a phenomenological analysis of the physical problem and the effects of the governing parameters, which in turn enables a more systematic design tool for soil-aeration systems. We present solutions for the air-pressure distributions resulting from harmonic (sinusoidal) or square-wave (step-pulse) air injection into infinite and semi-infinite (with an atmospheric pressure at the soil surface) soil domains. The approach presented here suggests using exact (analytical) solutions to extreme (unrealistic) conditions, setting the boundaries of the physical span of the flow problem. The simplified analytical solutions, based on assuming instantaneous water relaxation or, alternatively, assuming stagnant water are used to analyze the effects of different air-injection modes (constant, harmonic, and step-pulse) on the air-pressure distribution and radius of influence (ROI) for different values of the governing parameters: source depth, cycle period, and the soil's air-conductivity parameters. For long cycle periods, square-pulse and harmonic air injection can increase the maximal ROI compared with that resulting from steady air injection.

Soil aeration by air injection is a common practice in contaminated soil remediation (Nadim et al., 2000; Rathfelder et al., 2000) and has also been used to counteract O₂ deficiencies in agriculture (Melsted et al., 1949; Busscher, 1982; Vyrlas and Sakellariou-Makrantonaki, 2005; Niu et al., 2013; Lee et al., 2014; Li et al., 2015; Ben-Noah and Friedman, 2016). The latter, also termed soil hypoxia, is especially prevalent in intensively irrigated, fine-textured soils (Friedman and Naftaliev, 2012; Ben-Noah and Friedman, 2018) as well as in petroleum-contaminated ones (Dupont, 1993; Hinchee, 1994) negatively affecting root and shoot growth and other metabolic and yield parameters (Russell, 1952, 1973; Grable, 1966; Black, 1968; Stolzy, 1974; Smith, 1977; Armstrong, 1980; Stolzy et al., 1981; Gliński and Stepniewski, 1985; Drew, 1992; Amthor, 2000; Bhattarai et al., 2005; Stepniewski et al., 2005; Stepniewski and Stepniewska, 2009; Ben-Noah and Friedman, 2018). Nevertheless, despite its prevalence and well-established use in environmental practice, the air-injection technique has scarcely been used in agriculture. This is even more surprising if one considers the relatively simple and inexpensive application of air injection in plots where subsurface irrigation systems already exist, and that air injection was found useful in aerating the rhizosphere (Bhattarai et al., 2006; Ben-Noah and Friedman, 2016). This reluctance may be due to the lack of a simple and practical tool for the design of an efficient aeration system (Ben-Noah and Friedman, 2018).

Many studies have demonstrated improved air distribution using pulsed (periodic) air injection below (Shah et al., 1995; Johnson et al., 1999; Kirtland and Aelion, 2000; Heron et al., 2002; Yang et al., 2005; Baker and Benson, 2007; Balcke et al., 2009; Ben Neriah and Paster, 2017) and above (Rathfelder et al., 2000) the water table. The positive effects of step-pulsed or harmonically changing air-injection rates have been attributed to enhanced...
Air flow in partially saturated soils is a complex process because of the inherent interactions of viscous, capillary, and buoyant (gravity) forces driving air and water. This process is also subject to other phenomena, such as pressure-saturation hysteresis and soil heterogeneity and anisotropy. The modeling of two-phase flow is mostly restricted to numerical solutions, which require long computation times, may be impractical for most common uses, and have limited predictive capabilities.

Extremum states, such as steady air flow (Philip, 1998; van Dijke and van der Zee, 1998; Ben-Noah and Friedman, 2015) or the assumption of instantaneous water relaxation (IWR) suggested and tested here enable the description of the physical bounds of the air pressure and content distributions using simple analytical solutions of single-phase air flow.

We discuss the effects of cyclic air injection on the aeration of partially and completely water-saturated soils. Using analytical solutions, the models proposed here delineate the plausible range of air contents and pressures under the assumption of instantaneous water relaxation. This assumption enables a description of single-phase air flow with analytical solutions and avoids the complexity of the numerical solutions for two-phase flow, such as those of van Dijke et al. (1995), Silin et al. (2009), and Pasquier et al. (2017). The main suppositions of the proposed approach were corroborated using a numerical, one-dimensional, two-phase-flow model. The effects of system operation parameters, such as square vs. harmonic waves, and cycle period are analyzed with respect to the IWR-evaluated radius of influence and aerated soil volume. Furthermore, the effects of the soil water content (background potential) and water table on the attainable bounds of the air pressure and content distributions are discussed.

Analysis of Periodic Air Injection into a Partially Saturated Soil

We address the problem of a point source located within the soil volume and emitting air at a periodically (harmonic or square-wave) changing discharge rate by presenting solutions of air injection into infinite and semi-infinite soil domains while addressing the effects of an atmospheric soil surface or a water table on the air-pressure distribution. The different cases are depicted in Fig. 1.

Model Assumptions

During transient air injection into partially saturated soil, i.e., a soil whose pores are filled with both water and air, fluctuations in air pressure will result in water-pressure oscillations and, consequently, in the redistribution of the aqueous (complementary to air) phase. Therefore, to solve the problem of transient air injection, one must address the complex coupled equations of mass and momentum transfer of both air and water (Bear and Logeat, 1986; Gray and Hassanizadeh, 1998).

According to the Darcy–Buckingham law, water flow will take place only when a gradient in the total water head \( \langle \phi_w \rangle [L] \) prevails. The total water head is the sum of the water pressure \( h_w = p_w/\rho_w g \) [L] and elevation \( z \) [L] heads:

\[
\phi_w = h_w + z
\]

where \( p_w \) [M L\(^{-1}\) T\(^{-2}\)] is the water pressure, \( \rho_w \) [M L\(^{-3}\)] is the water mass density, \( g \) [L T\(^{-2}\)] is the gravitation constant, and \( z \) is the height above a reference level. When the gradient of the total water head declines to zero, hydrostatic conditions prevail. Under these conditions, any ongoing air flow does not change the air (or water) contents. In other words, the air–water distribution is bounded by that resulting from assuming instantaneous water relaxation to hydrostatic conditions. The IWR assumption implies that the water instantly reaches its hydrostatic distribution in response to the transient air-pressure fluctuations. This hydrostatic water distribution
is not that corresponding to the characteristic capillary pressure–saturation (water-retention) curve but rather a distribution that is affected by the air-pressure gradient. This assumption better applies to the physical conditions of short characteristic flow times for water ($k_w = 4/\alpha_w k_w$) [T], defined as the ratio between the capillary length ($\alpha_w$) [L] and the derivative of the water hydraulic conductivity ($K_w$) [L T$^{-1}$] with respect to the volumetric water content ($\theta_w$) ($k_w = dK_w/d\theta_w$) [L T$^{-1}$] (“characteristic velocity”), and of low-frequency air injection (long cycle period, $t_c$ [T]).

Air density, according to the ideal gas law, is proportionally dependent on the air pressure. Since the air pressure, for most air-discharge rates and soil air conductivities, deviates only slightly from atmospheric pressure (as seen below), disregarding the effect of air compressibility is justified.

The water and air phases are coupled by their volumetric contents, which comprise the soil porosity, and by the capillary head ($h_c$ [L]), i.e., the difference between the air-pressure and water-pressure heads:

$$h_c = h_a - h_w = h_a - \phi_w + z \quad [2]$$

Both hydraulic (water) and air conductivities are strong functions of the volumetric water and air contents, and consequently of the capillary head. Gardner (1958) suggested an exponential function relating the hydraulic conductivity, $K_w$ [L T$^{-1}$], to the capillary head:

$$K_w(h_c) = K_w^0 \exp \left[-\alpha_w (h_c - h_{\text{entry}}) \right] \quad [3]$$

where $K_w^0$ [L T$^{-1}$] is the hydraulic conductivity of a water-saturated soil and $h_{\text{entry}}$ [L] is the air-entry value. For simplification purposes, we disregard the latter here. Similarly, Philip (1998) suggested an exponential function relating air conductivity, $K_a$ [L T$^{-1}$], to the capillary head:

$$K_a(h_c) = K_a^0 \exp \left(\alpha_a h_c \right) \quad [4]$$

where $\alpha_a$ [L$^{-1}$] and $K_a^0$ [L T$^{-1}$] are positive soil parameters. The $K_a^0$ parameter is the minimal air conductivity value, not to be confused with the more common and bona fide parameter of maximal air conductivity ($K_a^s$ [L T$^{-1}$]) of an oven-dried soil. Using this convex exponential function to describe the dependence of air permeability on the capillary head allows analytical solutions for steady air injection below (Philip, 1998) and above (Ben-Noah and Friedman, 2015) a water table. Another advantage of the proposed air conductivity–capillary head function is the ensuing dependence of steady air flow on a single soil characteristic parameter ($\alpha_a$), which is the reciprocal of the air capillary (sorptive) length and a measure of the ratio between the effects of buoyancy and capillary forces on air flow (Ben-Noah and Friedman, 2015). This convex (positive curvature) function is expected to serve as a reasonable approximation at medium-range capillary pressures and be less applicable in near-water-saturated or dry soils (Ben-Noah and Friedman, 2015).

The main advantage of this suggested air-conductivity function (Eq. [4]) is that it allows analytical solutions in terms of the matrix flux potential (MFP), $\varphi$ [L$^2$ T$^{-1}$], defined for air as

$$\varphi = \frac{K_a^0}{\alpha_a} + \int_0^{h_c} K_a \, dh_a = \frac{K_a^0 \exp (\alpha_a h_c)}{\alpha_a} = K_a \quad [5]$$

Using the Kirchhoff transformation (the MFP), the linearized flow (continuity) equation (originally proposed for water flow) is (Philip, 1968; Warrick, 1974)

$$\frac{\alpha_a}{K_a} \frac{\partial \varphi}{\partial t} = \nabla^2 \varphi - \alpha_a \frac{\partial \varphi}{\partial z} \quad [6]$$

where $\nabla^2$ is the Laplacian operator, $z$ is the vertical axis taken positive upward, and $k_a = dK_a/d\theta_a$ [L T$^{-1}$], where $\theta_a$ is the volumetric air content.

The assumption that the air conductivity or permeability is a linear function of the air content, i.e., that $k_a$ is constant, is supported by some documented measurements (Clayton, 1999; Moldrup et al., 2001) but not by others (Selker et al., 2007; Assouline et al., 2016). In principle, this assumption is usually considered applicable strictly for flow in a relatively small range of air contents, as $k_a$ is expected to vary for different air-content ranges, with higher values of $k_a$ at low air contents. Yet, the above-mentioned literature data and our own measurements (not published) support its validity and usefulness even across a wide range of air contents.

Assuming that the Darcy–Buckingham law is applicable for air flow in partially saturated soils, the linearized flow equation (Eq. [6]) can be transformed from water flow to air flow by simply reversing the $z$ direction to be increasing upward (as done for steady air flow by Ben-Noah and Friedman, 2015). The conjunction of Darcy’s law and the definition of the capillary head (Eq. [2]) gives

$$q_a = -K_a \nabla h_a = -K_a \nabla (h_c + h_w) = -K_a \nabla (h_c + z - \phi_w) \quad [7]$$

where $q_a$ [L T$^{-1}$] is the volumetric air flux.

Conditions of hydrostatic water ($\phi_w = \text{constant}$) are equivalent to assuming a buoyancy force acting on the air, equal in magnitude and opposite in direction to the effect of gravitation on the water. As already noted, assumption of a constant total water head is at the heart of the IWR solutions.

Harmonic Air Injection from a Point Source into an Infinite Soil Domain

The general solution ($\varphi_\infty$) of Eq. [6] for a point source buried in an infinite domain and emitting at an arbitrary rate $Q(t) = Q_0 f(t)$ with a constant $Q_0 [L^3 T^{-1}]$ was given by Carslaw and Jaeger (1959):

$$\varphi_\infty = \frac{Q_0}{2} \exp \left(0.5 \alpha_a z \right) \frac{\exp \sqrt{\frac{1}{4} \left( r^2 + z^2 \right)}}{\sqrt{r^2 + z^2}} \times \frac{\exp \left(\frac{1}{4k_a} \frac{\sqrt{r^2 + z^2}}{r} \right)}{\left(4k_a \eta^2 \right)^1/2} \exp \left(\frac{\alpha_a \left( r^2 + z^2 \right)}{4k_a \eta^2} \right) \left[ \frac{\exp \left( \frac{\alpha_a \left( r^2 + z^2 \right)}{16 \eta^2} \right)}{16 \eta^2} \right] d\eta \quad [8]$$
Introducing the dimensionless variables:

\[
\frac{R}{r} = \frac{Z}{\phi_w} = \frac{H_w}{2}
\]

\[
\rho = \left( R^2 + Z^2 \right)^{1/2}
\]

\[
T = \frac{\alpha_w k}{4}
\]

and

\[
\Phi = \frac{8\pi \varphi}{\alpha_w Q_0}
\]

the solution for the nondimensional (ND) MFP resulting from a periodic harmonic variation in the air-discharge rate

\[
Q(t) = Q_0 + Q_0 \sin(\omega t)
\]

where \(\omega = 2\pi t_c [T^{-1}]\) is the radian frequency (angular velocity) and \(t_c [T]\) is the cycle period during which \(Q\) oscillates in the range of \(0 \leq Q \leq 2Q_0\), for a buried point source located at \(Z = 0\), was given by Communar and Friedman (2014), where it was described as

\[
\Phi_{\infty} = \frac{1}{2} \left( \Phi_C + \Phi_P \right)
\]

with the superscripts \(C\) and \(P\) denoting the constant and periodic flux terms, respectively. The MFP resulting from the constant flux term (\(\Phi_C\)) is (Philip, 1968)

\[
\Phi_C = \frac{\exp(Z - \rho)}{\rho}
\]

and the periodic MFP term (\(\Phi_P\)) is (Communar and Friedman, 2014)

\[
\Phi_P = \frac{\exp(Z - \mu_1 \rho)}{\rho} \sin(\omega_0 T - \mu_2 \rho)
\]

where \(\omega_0 = 4\omega/\alpha_w k_z\) is the dimensionless frequency and the coefficients \(\mu_1\) and \(\mu_2\) are

\[
\mu_1 = \frac{1}{2} \sqrt{\left( 1 + \omega_0^2 \right) + 1}
\]

\[
\mu_2 = \frac{1}{2} \sqrt{\left( 1 + \omega_0^2 \right) - 1}
\]

The ND vertical air flux \([U_z = 8\pi u_z/(\alpha_z^2 Q_0)\), where \(u_z [L T^{-1}]\) is the dimensionless vertical flux] according to Darcy’s law and the definition of the MFP is (Philip, 1968)

\[
U_z = \frac{1}{2} \frac{\partial \Phi_{\infty}}{\partial Z}
\]


\[
U_z^{\infty} = \frac{\exp(Z - \rho)}{4\rho} \left[ 1 + \frac{Z}{\rho^2} + \exp[\rho(1 - \mu_1)] \right] \times \left[ \sin(\omega_0 T - \mu_2 \rho) \left( 1 + \mu_1 \frac{Z}{\rho} + \frac{Z}{\rho^2} \right) + \mu_2 \frac{Z}{\rho} \cos(\omega_0 T - \mu_2 \rho) \right] + \mu_2 \frac{Z}{\rho} \cos(\omega_0 T - \mu_2 \rho)
\]

The ND radial air flux \([U_r = 8\pi u_r/(\alpha_r^2 Q_0)\), where \(u_r [L T^{-1}]\) is the dimensional radial flux] is (Philip, 1968)

\[
U_r = -\frac{1}{2} \frac{\partial \Phi_{\infty}}{\partial R}
\]

resulting in

\[
U_r^{\infty} = \frac{\exp(Z - \rho)}{4\rho} \left[ 1 + \frac{Z}{\rho^2} + \exp[\rho(1 - \mu_1)] \right] \times \left[ \sin(\omega_0 T - \mu_2 \rho) \left( 1 + \mu_1 \frac{R}{\rho} + \frac{R}{\rho^2} \right) + \mu_2 R \cos(\omega_0 T - \mu_2 \rho) \right] + \mu_2 R \cos(\omega_0 T - \mu_2 \rho)
\]

and the magnitude of the dimensional air flux, \(q^\infty [L T^{-1}]\), is

\[
q^\infty = \frac{2}{8\pi} \sqrt{U_r^{\infty} + U_z^{\infty}}
\]

The magnitude of the air flux is a better criterion for soil aeration and a better measure of aeration effectiveness than the commonly used air pressure or even air content (or MFP) criteria. However, a practical method for in situ assessment of soil air fluxes is still lacking.

The distributions of the capillary and air-pressure heads can easily be obtained based on the definitions in Eq. [2], [4], and [5]:

\[
b_c^{\infty} = \frac{1}{\alpha_a} \ln \left[ \frac{\alpha_a^2 Q_0 (\Phi_C + \Phi_B)}{8\pi K_a^{0.5}} \right]
\]

\[
b_a^{\infty} = \frac{1}{\alpha_a} \ln \left[ \frac{\alpha_a^2 Q_0 (\Phi_C + \Phi_B)}{8\pi K_a^{0.5}} \right] + \phi_w - z
\]

where \(\Phi_B\) is the background MFP (defined in Eq. [37] and discussed below), added to \(\Phi_{\infty}\) to address the hydrostatic water-pressure distribution (according to the characteristic water-retention curve) above a water table located at \(Z = H_w\) (Fig. 1) (Ben-Noah and Friedman, 2015), which has a negative value for air injection into unsaturated soil. We do not address the relevant issue of capillary-pressure–water-content hysteresis and assume that the capillary pressure is a unique function of the water (or air).
content. The history independence in the context of two-phase flow was discussed by Erpelding et al. (2013).

The amplitudes of the ND MFP (without the background MFP term) and ND air-pressure heads ($\alpha h / 2$), depicted in Fig. 2, increase with the cycle period ($T_c$) in the whole soil domain. The air-pressure heads in this figure (Fig. 2) were derived assuming $H_w = -1$, $K_a^0 = 0.06 \text{ cm s}^{-1}$, and $\alpha_a = 0.08 \text{ cm}^{-1}$ for the background MFP (needed for the evaluation of the air-pressure head). For short cycles ($T_c = 0.1$), the MFP and pressure distribution hardly change with time, while for long cycles ($T_c = 100$) the MFP and air approach their steady-state distribution. This suggests that for long cycles, the IWR assumption can be used as an approximation of the “real” solution and not just as extrema bounds. Moreover, for $T_c \to \infty$, both the “real” (MFP and air pressure) and the here-proposed (assuming IWR) distributions converge at each time ($T$) to the steady-state distribution resulting from air injection at a momentary rate of $Q(T)$ (analyzed by Philip [1998] and Ben-Noah and Friedman [2015]). For the case given here (Eq. [11]), the solution oscillates between the solution for $Q = 0$ (at $T = 0.75T_c$) and that for $Q = 2Q_0$ (at $T = 0.25T_c$). Thus, when $T_c \to \infty$, the MFP amplitude converges to the ratio between the maximum air-discharge rate ($2Q_0$) and the mean discharge rate ($Q_0$, i.e., a ratio of 2; Fig. 3b). For a high-frequency injection ($T_c \to 0$), the distributions of the air pressure and MFP converge to the steady-state distributions for constant air injection at a rate of $Q_0$. This is somewhat counterintuitive in the sense that despite the poor physical applicability of the assumption of instantaneous water relaxation (corresponding to assuming infinite water conductivity, $K_w \to \infty$) for air injection under short cycle periods, the actual (“real”) air status converges to that of the confining solution. This happens because of the reduced amplitude of the confining extrema. Another explanation is that the high-frequency oscillations of air pressure around a mean value near the source do not affect the air pressure away from the source. This means that the water is practically stagnant, and consequently, the air content is constant. According to this conclusion, under conditions of dry soil (or long characteristic water-flow times, $t_w = 4/\alpha_w K_w$) and high-frequency oscillations (small $T_c$), the use of cyclic air injection will not improve soil aeration in comparison to constant air injection.

![Fig. 2. Equal nondimensional (ND) air-pressure head (isobars, black) and equipotential (matric flux potential [MFP], red) lines for injection of air from a buried point source in an infinite, partially saturated soil domain at different stages of the harmonic air-injection cycle (left to right) and for different cycle periods (increasing downward). The pressure distribution (but not the MFP) was drawn accounting for a background MFP corresponding to a ND total water head of $H_w = -1$, soil parameters $\alpha_a = 0.08 \text{ cm}^{-1}, K_a^0 = 0.06 \text{ cm s}^{-1}$ and mean air-discharge rate of $Q_0 = 100 \text{ L h}^{-1}$.](image-url)
Test of the Instantaneous Relaxation and Stagnant Water Approximations

Normally, for most soil textures and water contents, the characteristic flow time of water \( \left( \frac{4}{\alpha_w k_w} \right) \) is much larger than that of air \( \left( \frac{4}{\alpha_a k_a} \right) \) because of the water’s 50-fold higher dynamic viscosity. Therefore, the “real” (coupled air–water flow) distributions of air–water contents and pressures are expected to be closer to those resulting from a constant-rate air injection at the mean discharge rate \( (Q_0) \), referred to here as the stagnant water assumption, than to the IWR distributions (evaluated and analyzed here). The difference between the instantaneous water relaxation and stagnant water assumptions can be clarified with regard to their different effects on the total water head \( (\phi_w) \). Under the former, the total water head \( (\phi_w) \) remains constant (in space and time); in contrast, under the latter, the total water head oscillates (spatially and temporally), keeping the capillary pressure head \( (h_c) \) constant. Hence, the stagnant water assumption suggests that the air-pressure oscillations do not change the water–air content distribution. This means that the induced water-pressure gradient does not cause water fluxes (in contradiction to Darcy’s law, or alternatively, assuming zero water conductivities, \( K_w \rightarrow 0 \)). In this case, the MFP is simply the constant-rate air-injection flux MFP \( (\Phi_{\infty}) \) given in Eq. [12]. For the sake of simplicity, we used one-dimensional (1D), vertical flow models to test the consequences of the suggested approach in a qualitative manner.

Two-Phase Flow Model

We constructed a 1D, two-phase flow model using the well-known, mass conservative, finite difference, head-based Richards equation scheme of Celia and Binning (1992), described in the appendix. In this model, we used the nonlinear expressions for the hydraulic conductivity \( (K_w) \), air conductivity \( (K_a) \), water capacity \( (C_w) \), and water retention curve \( [\theta(h_c)] \) consistent with the assumption \( dK_a/dq_a = k_a \) and with Eq. [3] and [4]. The boundary conditions for the air flow are harmonic air flux \( (q = q_0[1 + \sin(\omega t)]) \) at the bottom and a constant atmospheric pressure at the upper boundary. The boundary conditions for the water were set to be constant pressure heads at both ends, \( h_w(z = 0) = \phi_w \) at the bottom and \( h_w(z = l) = \phi_w - l \) at the top, where \( l \) is the length of the soil domain. Changing the upper water boundary condition to a “no flow” boundary had a negligible effect on the distributions of air pressure and MFP (not presented). The two-phase-flow numerical solution was compared with both stagnant water and instantaneous water relaxation analytical solutions for 1D air flow described below.

Stagnant Water Model

The capillary head (and MFP) distributions resulting from the stagnant water assumption are obtained from the solution corresponding to the constant, mean air flux \( (q_0) \). The solution for steady, 1D air flow generated by a constant flux from the bottom...
of a finite soil domain with prescribed atmospheric pressure at its surface can be derived from Darcy’s law (Eq. (7)) simply by the separation of variables:

$$\int_{z}^{h} dz = -\int_{z}^{h} \frac{K_a(h_z)}{K_a(h_z) - g_0} db_c$$ [22]

Introducing Eq. [4] for the air conductivity gives

$$b_c^\text{stag} = -\frac{1}{\alpha_a} \ln \left( \frac{g_0 \exp[\alpha_a (l-z)] + K_a \exp[\alpha_a (l-\phi_w)] - g_0}{K_a \exp[\alpha_a (l-z)]} \right)$$ [23]

$$0 \leq z \leq l$$

The dimensional MFP can be inferred from the capillary pressure, as for three dimensions, using Eq. (5); however, unlike the three-dimensional (3D) MFP definition (Eq. (9)), the one-dimensional ND-MFP is

$$\Phi^{1D} = \frac{\phi^{1D} \alpha_a}{g_0}$$ [24]

Thus, the stagnant water solution in terms of the ND MFP is

$$\Phi^{1D}_{\text{stag}} = 1 + \frac{K_0}{g_0} \exp[2(Z - H_w)] - \exp[2(Z - L)], 0 \leq Z \leq L$$ [25]

where \( L = \alpha l/2 \).

Under the stagnant water assumption, both MFP and capillary pressure are constant in time. This means that the air flux is uniform in the entire soil domain, causing the air pressure to oscillate simultaneously without a time shift but rather with an amplitude, dumped with distance from the source, throughout the whole domain. The air pressure head induced by a harmonic source at the bottom of the soil domain is according to Darcy’s law:

$$b_c^\text{stag} = \left[ g_0 + g_0 \sin(\omega t) \right] \int_{z}^{h} dz$$

$$= \frac{1 + \sin(\omega t)}{\alpha_a} \left[ \alpha_a (l-z) + \ln \left( \frac{K_a \exp[\alpha_a (z-\phi_w)]}{K_a \exp[\alpha_a (l-\phi_w)]} \right) - \frac{g_0 \exp[\alpha_a (z-l)] + g_0}{K_a \exp[\alpha_a (l-\phi_w)]} \right]$$ [26]

$$0 \leq z \leq I$$

Instantaneous Water Relaxation Model

Bakker and Nieber (2009) introduced a solution for a harmonic flux prescribed at the boundary of a semi-infinite 1D soil domain, described as a superposition of solutions for a constant and a cyclically oscillating source. Using the complex form of the harmonic air flux boundary condition at the bottom of the domain (in their case it was harmonic water influx at the upper, soil surface boundary) and a zero pressure head at infinity, they derived the following solution for the sinusoidal cycle MFP (without the component of the MFP caused by the constant flux)

$$\Phi^{1D}_{\text{relax}} = \text{imag} \left[ A \exp(2aZ + i\omega_0 T) \right], 0 \leq Z$$ [27]

where the constants \( a \) and \( A \) are

$$a = \frac{1}{2} - \frac{1}{2} \left( 1 + i\omega_0 \right)^{1/2}$$

$$A = \frac{1}{1-a}$$

(In the case of a cosine source, the real part of Eq. (27) should be taken rather than its imaginary part).

Based on this solution, the upper boundary of zero air pressure can be satisfied simply by adding an imaginary sink of weighted strength (Philip, 1989). Then the constant \( A \) should be adjusted to preserve the bottom boundary condition of the harmonic air source. Thus, the solution for a sinusoidal air source injection into a finite soil domain with atmospheric pressure prevailing at its surface is

$$\Phi^{1D}_{\text{relax}} = \text{imag} \left[ \exp(i\omega_0 T) \right] \times \left[ \exp \left\{ Z - Z(1 + i\omega_0) \right\} - \exp \left\{ Z - 2L(1 + i\omega_0) \right\} \right]$$ [29]

$$1 - a - a \exp \left\{ -2L(1 + i\omega_0) \right\}$$ $0 \leq Z \leq L$

Comparison of the Three Models

As seen in Fig. 3a the amplitude of the “real”, two-phase flow, MFP distribution is bounded by that of the solution of the IWR assumption, supporting the approach recommended here. In addition, as anticipated, for short cycles, the two-phase flow solution is satisfactorily approximated by the stagnant (steady-state) solution, and it is becoming similar to the instantaneous relaxation solution with increasing cycle period (\( T_c \)). The MFP under the IWR assumption approaches its steady-state value at a much smaller \( T_c \) than the two-phase solution. While the temporal wavelengths of the MFP oscillations in the entire domain are expected to be the same (\( T_c \)), the ND time shift, \( T_c \) (i.e., the ND time difference between maximum MFP at a given location and the ND time of maximum source-discharge rate, 0.25 for a sine source) of the instantaneous relaxation solution diverges from that of the two-phase flow solution. The length of the \( T_c \) is a function of both location (\( Z \)) and cycle period (\( T_c \)), increasing with distance from the source and with decreased cycle period (Fig. 3b).

The confinement of the amplitude of the “real” solution by that of the IWR solution and the other conclusions here are expected to apply for 3D air flow as well. We now proceed with analyzing 3D air flow induced by injection from a point source using the IWR assumption.

Air Pressure and Air Content Distribution around a Point Source

The air-pressure head resulting from harmonic air injection from a point source (accounting for a background, hydrostatic capillary-pressure distribution) is lowest above the source and highest below it (Fig. 4). The air pressure in the radial direction (at the same depth as the source) is a bit closer to that found at a similar
distance above the source. On the other hand, the MFP (or air content, since the volumetric air content, \( q_a \), is proportional to \( K_a \) and to the MFP, \( K_a/\alpha_a \)) is higher above the source and lowest below it, even without the effect of the background potential. The ND time shift is identical (0.13 for this case of \( T_c = 1 \)) for all three locations.

**Harmonic Air Injection at an Arbitrary Depth below the Soil Surface**

We consider the problem of a point source located in a semi-infinite soil domain at a distance of half ND length, above (blue), below (black), and in the radial direction (red) from the source. The ND cycle period is \( T_c = 1 \), total water head is \( H_w = -1 \), the soil parameters are \( \alpha_a = 0.08 \text{ cm}^{-1}, K_a = 0.06 \text{ cm s}^{-1} \), and the mean air-discharge rate is \( Q_0 = 100 \text{ L h}^{-1} \).

\[ \Phi_{\infty,\text{pulse}}(Z) = \frac{\exp(Z - \rho)}{2\rho} + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\exp(Z - \mu_m \rho) \sin(\omega_m T - \mu_m \rho)}{(2m-1)\rho} \]

where \( \mu_m = (2m-1)\omega_m \) and \( \frac{\omega_m}{\alpha_a} = \frac{\omega_m^2}{2m} \).

The solution for step-pulse air injection below an atmospheric soil surface at a height \( z_0 \) above the point source is formulated using the same above-described principle of weighted imaginary sinks (Eq. [30]). Figure 6 shows the distributions of the MFP and air-pressure head at several times (left to right) along the cycle and for different cycle periods (increasing from top to bottom), for the case of 50% duty, step-pulse air injection from a point source located at a ND depth of \( Z_0 = 1 \) below an atmospheric soil surface.

**Square-Wave (Step-Input) Air Injection**

Unlike the harmonic air injection described above, which requires special equipment, square-wave (step-input) air injection is more practical and easier to implement. Cyclic-step air injection with equal on and off times (50% duty cycle) can be represented as a Fourier sine series of the aforementioned harmonic air injection (Communard and Friedman, 2014):

\[ Q(t) = Q_0 + Q_0 \sum_{m=1}^{\infty} \frac{4 \sin(\omega_m t)}{\pi (2m-1)} \]

comprised of only odd-number harmonics: \( \omega_m = (2m-1)\omega \). Thus, the MFP resulting from cyclic step-input air injection into an infinite soil domain is:

\[ \Phi_{\infty,\text{pulse}}(Z) = \frac{\exp(Z - \rho)}{2\rho} + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\exp(Z - \mu_m \rho) \sin(\omega_m T - \mu_m \rho)}{(2m-1)\rho} \]

where \( \mu_m \) is determined by Eq. [14], replacing \( \omega \) with \( \omega_m \).
and using the same values as in Fig. 2 and 5 for evaluating the air pressure ($H_w = -1$, $K_a^0 = 0.06$ cm s$^{-1}$, $a = 0.08$ cm$^{-1}$, and $Q_0 = 100$ L h$^{-1}$). As for harmonic air injection, the "real" (coupled air–water) solution is expected to be bounded by that assuming instantaneous water relaxation. Moreover, as before, this assumption is more applicable for long cycle periods for which the solution (MFP and air-pressure distributions) oscillates between the distributions for no air injection and constant-rate air injection at a discharge rate of $2Q_0$. For short cycle periods, the solution (similarly to the case of harmonic injection) converges to the solution of constant-rate air injection at a rate of $Q_0$. This means that using very short (compared with $t_w$) pulses will not improve soil aeration compared with harmonic or even constant-rate air injection.

Aerated Volume and Radius of Influence

The $O_2$ consumption of intensive agriculture soils is about $10 \text{ g } O_2 \text{ m}^{-2} \text{ d}^{-1}$ (Friedman and Naftaliev, 2012; Ben-Noah and Friedman, 2018), while atmospheric air contains about $250 \text{ g } O_2 \text{ m}^{-3}$. This means that the gaseous phase of soil, assumed to be fresh, $21\% O_2$ air, of a root zone of, e.g., 50-cm depth and a mean volumetric air content of 0.15 contains about $20 \text{ g } O_2 \text{ m}^{-2}$ soil, i.e., it should be replaced every 2 d. In other words, an air-discharge rate of $0.04 \text{ m}^3 \text{ m}^{-2} \text{ d}^{-1} (1.6 \text{ L h}^{-1} \text{ m}^{-2} \text{ soil})$ is potentially enough to supply the soil $O_2$ demand. This leads to the conclusion that the desired air distribution, producing a wide spread, rather than the required discharge rates, is the main limiting factor of this soil-aeration practice.

Affected by buoyancy and the atmospheric pressure at the soil surface, the air-flow direction is mainly upward. Therefore, one of the most important aspects in evaluating an efficient soil-aeration system is to determine the air’s horizontal spread. For practical reasons, the measurable pressure or potential (air content), rather than the air flux, is usually used as a quantifier of the radius of influence (ROI) (McCray and Falta, 1996). According to this rationale, we can define aeration criteria such as the maximum soil volume (along the injection cycle) enclosed by an arbitrary MFP reference value, which should be large enough to avoid spatial repetitions in the soil domain at any given time. For example, using zero as the MFP reference value may lead to ambiguity because a surface of MFP = 0 is not unique in the soil domain subjected to harmonic air injection.
We use here the arbitrary value of 0.1 for the reference ND MFP to define the ND aerated soil volume. For a more general analysis of the effects of source depth and frequency, it is sufficient to address only the ND MFP reference value. The ND aerated soil volume is thus

\[ V = \pi \int_{-\infty}^{Z_0} R(\Phi_{as} = 0.1)^2 \, d\xi \]  \[ \text{[34]} \]

and the ND ROI is defined here as the radius of a cylinder with the same volume as that of the aerated soil volume and a height of \( Z_0 \):

\[ \text{ROI} = \frac{V}{\pi Z_0} \]  \[ \text{[35]} \]

The ND MFP is independent of the mean air-discharge rate \( Q_0 \), whereas the dimensional MFP is proportional to \( Q_0 \) (Eq. [9]). The appropriate dimensional \( Q_0 \) \([L^3 T^{-1}]\) can be set based on the maximal dimensional ROI and the expected areal \( O_2 \) consumption rate:

\[ Q_0 = \frac{S_{O_2}}{C_{O_2}} \max \left( \frac{2 \text{ROI}}{\alpha_s} \right)^2 \]  \[ \text{[36]} \]

where \( S_{O_2} \) \([M L^{-2} T^{-1}]\) is the soil areal \( O_2 \) consumption and \( C_{O_2} \) \([M L^{-3}]\) is the atmospheric \( O_2 \) concentration.

For long cycle periods (approaching a quasi-steady state), the maximal ROI is also bounded by the value corresponding to steady air injection at a discharge rate of \( 2Q_0 \). As already noted, under short cycle periods, the solution tends to that of constant air injection at \( Q_0 \). The maximum value of the ROI for the case of square-wave air injection only slightly exceeds that of harmonic air injection for most cycle periods (\( T_c \), Fig. 7). However, the ROI of the constant air injection is significantly lower than the maximum ROI achieved with the two modes of periodic air injection (Fig. 7). These observations are in accordance with experimental results reporting increased removal efficiency of a volatile compound using step-pulse air injection compared with constant-rate air sparging (Ben Neriah and Paster, 2016, 2017). Increasing the source depth \( (Z_0) \) will increase the ROI only for long enough cycle periods, and vice versa—increasing the cycle period \( (T_c) \) will only increase the ROI for deep enough sources. For long \( T_c \) (right-hand side plots in Fig. 7), increasing the source depth \( (Z_0) \) from 0.1 (top plot) to 1 (middle plot) increases the ROI of the step-pulse air injection by a factor of almost 3, while further increase to \( Z_0 = 10 \) (bottom plot) increases the ROI only by a factor of <1.25. Interestingly, for the intermediate cycle period of \( T_c = 1 \) (the middle column), increasing the source depth \( Z_0 \) from 1 to 10 reduces the
maximum ROI of the cyclic air-injection modes. This is because the flow is strictly upward in most of the soil domain so that the horizontal and downward air flow near the source has little and declining (with increasing depth, \( Z_0 \), Eq. [35]) effect on the aerated soil volume (\( V \)). The limited effect of source depth on the ROI is caused by the dominating upward air flow due to buoyancy forces and, to a lesser extent, owing to soil-surface atmospheric pressure. For shallow sources and long cycle periods, the ROI of the periodic (step-pulse and harmonic) injection modes varies symmetrically around an average ROI resulting from constant air injection. The ND time shift of the ROI (\( T \), for which ROI is maximal minus 0.25) depends on both source depth (increases with \( Z \)) and cycle period (decreases with \( T_0 \)). For the case of step-pulse air injection, the optimal cycle period (\( T_c \)) can be obtained by evaluating (or measuring) the cycle period for which the air pressure at a reference point of interest practically reaches a plateau. For some applications (e.g., enhancing root respiration), the desired criterion should not be a maximized mean ROI along the injection cycle but rather a maximum ROI at the relevant times (e.g., daytime). Under these circumstances, the optimal cycle period should be set to a value that is independent of soil properties and source depth.

**Effect of Background Air–Water Distribution**

As mentioned above, the MFP given by Eq. [11] and [33] corresponds to the case of a water-saturated background soil (equivalent to a dry background soil for water flow). For the case of air injection into a partially saturated soil, a background MFP should be added, as done in all the air-pressure computations (but not for the MFPs) presented and discussed here. The background potential is also important for evaluating the distribution of air contents (and capillary pressures). The background potential represents the hydrostatic water distribution (corresponding to the water retention curve) above a water table located at \( Z = H_w \) (Ben-Noah and Friedman, 2015):

\[
\Phi_{B,\text{unsat}} = \frac{8\pi K_w^0}{\alpha_4 Q_0} \exp\left[2(Z - H_w)\right]
\]

This background potential is a consequence of the water distribution above a phreatic surface resulting from the equilibrium between capillary and gravitational forces. In terms of air distribution, the background potential is the distribution of capillary pressure, and the corresponding air content, resulting from an equilibrium between capillary and buoyancy forces. This background potential is unique in the sense that it does not cause any air flow, i.e., it does not affect the flux terms, Eq. [15] and [17], but only the air-pressure distribution. Adjusting Eq. [11] to its more general form containing this background gives

\[
\Phi_{\text{C-sat}} = \frac{1}{2} \left( \Phi_C - \Phi_B \right) + \Phi_B
\]
The relative effect of the background MFP increases in dry soils (low $H_w$) and for low air-discharge rates ($Q_0$). Under these conditions, the background MFP is much higher than the MFP induced by the air-injecting source. According to Eq. [37], the background potential decreases exponentially with increasing total water head ($H_w$, the ND depth of the water table). Despite this reduction in the MFP component, the air pressure (which corresponds to the MFP according to Eq. [21]) is expected to be lower in dry soils (low total water head) due to the larger air conductivities. Addressing a MFP without the background term ($\Phi_B$), as in Philip (1998), results in a MFP that is independent of the total water head, which in turn means (according to Eq. [21]) that the air pressure depends strongly (linearly at large distance from the source) on the total water head.

**Air Injection below a Water Table**

Step-pulse air injection is considered the best practice for groundwater remediation when using the air-sparging method (Johnson et al., 1993; Hinchee, 1994). This method uses injection of atmospheric air to remove volatile contaminants from the groundwater to the unsaturated zone and atmosphere; it is usually accompanied by an on-site treatment facility, such as soil-vapor extraction, which collects and treats the effluent-contaminated air (Johnson et al., 1993; Ahlfeld et al., 1994). In addition, the $O_2$-rich (21%) injected air increases the dissolved $O_2$ concentration and therefore the decomposition rate of the contaminants by aerobic microbial activity (Thomson and Johnson, 2000). Using step-pulse air injection increases the ROI by preventing the formation or allowing the closure of previously formed preferential vertical pathways ("fingers" or "chimneys") for air flow (Thomson and Johnson, 2000). Despite the limited ability of the continuum approach to describe air injection into a water-saturated medium, as discussed by Ben-Noah and Friedman (2015), or McCray (2000) for heterogeneous media, some general insights may still be derived from its application.

The definition of the MFP and the solutions presented above are restricted to air injection into the soil above the water table. To make use of these solutions in a more general context, it is necessary to apply one of the two assumptions associated with a physical contradiction: (i) adding a constant, minimum potential ($\Phi_0 = k_2^0/\alpha_2$), which is consistent with zero capillary head at water saturation but problematic due to a positive, finite, air conductivity at water saturation; or (ii) extending the background MFP in a continuous manner to below the water table, thus attaining a zero value at infinite depth below the water table and the same value of $K_a^0/\alpha_a$ (indicating air conductivity) at the water table.

These two background MFP extensions are depicted in Fig. 1. The latter extension (the second alternative) is equivalent to extending the capillary head to obtain negative values below the water table, which is irrational for hydrophilic soil particles in contact with air and water, and obviously not representative of a water-saturated soil (where the capillary head is not defined). Despite these limitations, the error of using the second alternative is confined because the air conductivity ranges only between the small value $K_a^0$ and zero, and thus it is expected to be less significant than the constant potential approximation (the first alternative). An alternative to adding a background MFP is extending the MFP definition (Eq. [5]), which can be simply obtained by implementing $-$ as the lower limit of integration and then dropping the constant $K_a^0/\alpha_a$ term. This formulation does not alter the solutions given above. Under the assumption of instantaneous water relaxation and within the continuum approach, the MFP distributions for air injection into an unsaturated soil or from a submerged source differ only in their background term. Nevertheless, the location of the air source with respect to the water table has a pronounced effect. For example, air injection from a ND depth of 1 below the water table compared with injection from a ND height of 1 above the water table (i.e., increasing the total water head from $H_w = -1$ to 1) increases the air pressure at a ND radial distance of 0.5 from the source by two orders of magnitude (not presented).

**Effect of System Parameters**

We next discuss the effect of the dimensional parameters $\alpha_a$, $k_2$, $Q_0$, and $z_0$ on soil aeration and on the design of efficient soil-aeration systems.

Under most circumstances, the soil parameters $\alpha_a$ and $k_2$ are given, while the air injection parameters, $Q_0$ and $t_2$, and the source depth, $z_0$, are free design parameters (up to some technical restrictions such as allowable air pressure and trencher or drilling equipment depth). Among these, the cycle period ($t_2$) seems to be the least restricted by technical limitations (usually it is the maximum operation time of compressors, i.e., cycle duty times, which is restricted).

The soil parameter $\alpha_a$ is the exponential constant in the relationship between the soil air conductivity and capillary head (Eq. [4], $\alpha_a^{-1}$ was termed above the air capillary length). In fine-texture soils with strong water retention, a given change in the capillary head will result in a less significant change in the air conductivity than in a coarse-texture soil. This means that $\alpha_a$ values of clayey soils are expected to be smaller than those of sandy soils. The counter-part parameter for water ($\alpha_w$, the reciprocal of the water capillary length) is a measure of the ratio between the effects of gravity and capillary forces on driving the water in the soil. Analogously, as stated above, $\alpha_a$ can be regarded as a measure of the ratio between the roles of buoyancy and capillary forces in driving the air. In accordance with Clayton’s (1998) findings, the dimensional ROI in small $\alpha_a$ (clayey) soil is expected to be larger than that in large $\alpha_a$ (sandy) soil because of the dominant capillary-pressure component in the water-pressure gradient compared with the buoyancy-force component. This conclusion may be counterintuitive because under most common practices, the air-discharge rate ($Q_0$) is not regulated. Therefore, in clayey soils (with small $\alpha_a$), due to the larger soil resistance to air flow (because of the smaller pores and higher water retention), the air-discharge rate is usually smaller and consequently so is the actual aerated soil volume. A prescribed (independent of $\alpha_a$) air-injection rate can be achieved using, for example, pressure-compensated drippers for air injection.
The parameter $K_a^0$ constitutes a physical contradiction, as the actual soil conductivity to air reaches zero at water saturation, and should, therefore, be regarded merely as a fitting parameter. Based on measurements of Clayton (1999) for capillary heads in the range of 10 to 40 cm H$_2$O in a coarse soil (his Case 4), $k_a$, $K_a^0$, and $\alpha_a$ are about 6.61 cm s$^{-1}$, 0.06 cm s$^{-1}$ (in water head units), and 0.08 cm$^{-1}$, accordingly. Based on these values, the cycle period ($t_c$) corresponding to an ND $T_c = 1$ (equal to the characteristic air-flow time, $4/\alpha_a k_a$) is about 8 s. According to the curves presented in Fig. 2, 5, 6, and the central plot of Fig. 7, for the spatial dimensions of this reference problem (a source depth of 25 cm), the source acts as a constant-rate source for a cycle period of 0.8 s ($T_c = 0.1$), while a 1.33-min cycle period ($T_c = 10$) results in a step-function MFP. These values seem reasonable for a coarse-texture soil. Using these parameters and Fig. 7, the range of ROIs for air injection at a depth of 25 cm (middle row in Fig. 7, $Z_0 = 1$) below the soil surface is between 31 cm for constant-rate air injection (or very short cycle periods) and 44 cm for long cycles using step-pulse or harmonic air injection. Communar and Friedman (2014) suggested a simple field method to determine the soil parameters ($k_a$ and $\alpha_a$) for the purpose of drip-irrigation design, based on measuring the time shift and amplitude of the capillary head (using a tensiometer) at a fixed depth below the source dripper. This simple procedure cannot be easily transformed for assessing the soil parameters for air flow because in this case much smaller, undetectable time shifts and air pressure amplitudes are expected (Fig. 8). The difference in the time shifts at a depth of 25 cm below the surface and right above the source, across large ranges of $\alpha_a$ (left plot), $k_a$ (middle plot), and $z_0$ (right plot), is just a few seconds. Furthermore, the maximal air pressure 25 cm above the source using the aforementioned coarse soil parameters is about 0.002 kPa. Using the parameters of a less, but still quite coarse-texture soil (Case 5 in Clayton, 1999) with permeability of 3.83 cm$^{-1}$ increases the maximum pressure by one order of magnitude (to about 0.02 kPa). As seen in Fig. 8, the time shift is a reciprocal function of the soil characteristic air velocity, $k_a$, a concave (negative curvature) function of the soil capillarity/buoyancy parameter, $\alpha_a$, and a linear function of (large enough) source depth, $z_0$. Whereas the time shift increases linearly with distance from the source, the air pressure drops steeply (right plot in Fig. 8). These results suggest that the use of a time shift and pressure amplitude at a given location for estimating the soil properties is more applicable below the source and in fine-texture and wet soils. The air pressure is logarithmically related to the air-discharge rate, $Q_a$ (Eq. [21]), while the dimensional MFP ($\varphi$) is proportional to $Q_a$ (Eq. [9]). This means that increasing the discharge rate, counterintuitively, will have a limited effect on the air pressure. The time shift is independent of the air-injection rate. The apparently different effect of $k_a$ on the time shift for the case of step-pulse air injection (decreasing asymptotically to a value of 1.89 s [= 0.25 $t_c$], middle plot in Fig. 8), compared with its effect for harmonic injection, is simply because of the way we define the time shift as the difference between the time (or average time in the case of a plateau) of maximum air pressure and half the time of the air injection ($t_c/4$).

The relationships between $\alpha_a$ and $\omega_a$ have never been investigated. Yet, as both parameters are strongly affected by the soil pore-size distribution and expected to be higher in coarse sands than in finer sands or clayey soils, one should expect some degree of correlation. The maximum, single-phase hydraulic ($K_a^i$) and air ($K_a^a$) conductivities $[LT^{-1}]$ are both proportional to the intrinsic soil permeability ($k_a$) $[L^2]$ and related according to

$$\frac{K_a^i}{K_a^a} = \frac{\gamma k_a/\mu_a}{\gamma k_a/\mu_a} = \frac{\mu_w}{\mu_a}$$

where $\mu_i$ $[ML^{-1}T^{-1}]$ is the dynamic viscosity of water ($i = w$) or air ($i = a$) and $\gamma$ $[ML^{-1}T^{-1}]$ is the water weight density. The use of $\gamma$ for both fluids unifies the use of the head terms in water length units. Bearing in mind that $k_a$ and $k_w$ are both derivatives of the hydraulic or air conductivity with respect to the volumetric water ($\theta_w$) or air ($\theta_a$) contents (respectively), and that $\theta_w$ and $\theta_a$ span the same range (fractions of the soil porosity, $\theta$), $k_a$ and $k_w$ can be roughly estimated by the ratio between the maximum hydraulic (or air) conductivity and the soil porosity ($k_a \approx K_a^i/\theta$, $i = a,w$). This means that the $K_a^i/K_a^a$ ratio (Eq. [39]) holds approximately for $k_a/k_w$. In general, because of the significantly lower dynamic viscosity of air (the ratio
in Eq. [39] is about 55 at 20°C, the characteristic flow time of air 
\((4/\alpha_{s}k_{a})\) in the unsaturated zone is significantly shorter than that of water 
\((4/\alpha_{w}k_{w})\). Therefore, equilibrium conditions for air flow are 
much more rapidly than those for water flow (van Dijke et al., 1995). Consequently, the water distribution during periodic air injection 
into a partially saturated soil is expected (for most reasonable cycle periods and water contents) to be closer to that corresponding to 
the constant-rate air injection (stagnant water) solution than to that of the IWR solution.

**General Discussion and Conclusion**

Soil aeration practice is severely underused, partly due to the lack of a simple and efficient design tool. Simple analytical 
solutions for various setups and conditions are a prerequisite for constructing such a tool. Using extremum states sets lower and 
upper bounds that confine the more elaborate, physically realistic solutions to different conditions.

The capability of the extremum bounds to characterize the real solution is highly dependent on the source’s cycle period. For short cycle times, the air content and pressure distributions 
diverge from the solution based on the stagnant water assumption. However, for extremely short periods, or at 
larger distances from the source, the IWR solution also converges to the stagnant water solution.

The air-pressure oscillations in the case of the stagnant water assumption should obey a diffusion type of flow equation with 
a known, spatially varying air-conductivity function 
\(K_{a}(R, Z) = \alpha_{a} \Phi(R, Z, Q_{0})\), Eq. [5]) corresponding to air injection at the mean 
and discharge rate \((Q_{0})\). Unfortunately, this equation, as far as we 
know, does not have an analytical solution and must be solved numerically. If solved numerically, the time-invariant air conductivity 
will probably significantly shorten the computation time and improve the stability of the numerical scheme by obviating 
the need for linearization of the conductivity function. In this case, the conductivity matrix needs to be inverted only once using 
conventional methods as lower-upper (LU) matrix decomposition.

Despite their limitations, analytical solutions derived by applying 
a continuum approach can be used for the preliminary design of aeration systems or for the design of a pilot-scale array. This approach 
acts as a compromise between using general, textbook-recommended parameters such as in USEPA (1994) or, alternatively, developing a 
detailed, 3D numerical model for coupled air and water flow. Such a model possesses a larger number of parameters, and its computational 
time is substantially longer, yet it has only “limited predictive capability at sites” (Thomson and Johnson, 2000).

Thanks to the linearity of the analytical solutions, a line source (such as an air-injection well) can simply be described as a superposition of point sources. However, for the case of air injection into a wet soil from a vertical line source, it should be 
taken into account that assuming a uniform discharge rate from all discrete point sources in the line is erroneous because more air will usually be discharged from the upper part of the injection well.

Although the present approach—assuming an extremum state of instantaneous water relaxation—does not solve the exact, 
coupled air–water physical problem, the resulting analytical solutions facilitate both the phenomenological description of air flow 
partially and initially fully water-saturated soils, as well as the practical engineering applications.

As a follow-up, we recommend a field study to evaluate the ROI (defined by a representative MFP value) under different biotic 
and abiotic conditions and for different soil properties. In addition, the soil’s \(\alpha_{s}\) and \(k_{s}\) parameters for different soil textures, structures, 
and pore-size distributions need to be better characterized.

**Appendix**

According to the continuum approach, the general partial differential equations governing the coupled flow of air and water are

\[
\frac{\partial (\rho_{a} \theta_{a})}{\partial t} + \nabla (\rho_{a} q_{a}) = S_{a}, \quad \theta_{a} = a, w
\]  

where the \(a\) and \(w\) subscripts stand for air and water, \(\rho_{a}\) is the density and \(\theta_{a}\), the volumetric content of phase \(\alpha\), \(q_{a}\) is the flux, and \(S_{a}\) \([\text{M} \text{L}^{-3} \text{T}^{-1}]\) is the source or sink term.

For one-dimensional flow of an incompressible fluid without a source or sink in the domain, and by substituting Darcy’s law 
(Eq. [7]) applicable for both air and water, Eq. [A1] reduces to Richards’ equations

\[
\frac{\partial \theta_{a}}{\partial t} - \frac{\partial}{\partial z} \left( K_{a} \frac{\partial h_{a}}{\partial z} \right) = 0
\]

\[
\frac{\partial \theta_{w}}{\partial t} - \frac{\partial}{\partial z} \left( K_{w} \frac{\partial \phi_{w}}{\partial z} \right) = 0
\]

These equations are coupled by the porosity being the sum of the volumetric contents \((\theta_{s} = \theta_{w} + \theta_{a})\) and by the definitions of the capillary pressure (Eq. [2]) and of the total water head (Eq. [1]). Introducing the water capacity \(C_{w}\) defined as \(-d\theta_{w}/d\phi_{w}\), enables the formulation of the head-based Richards equation for which Celia 
and Binning (1992) developed a finite difference, mass conserva 
tive scheme (disregarding the air compressibility and elastic storage features in their original scheme):

\[
C_{w} \frac{\theta_{w} + 1}{\theta_{w}} \frac{\delta \theta_{w}}{\delta t} = \frac{\partial}{\partial z} \left( K_{w} \frac{\partial \phi_{w}}{\partial z} \right)
\]

\[
C_{w} \frac{\theta_{a} + 1}{\theta_{a}} \frac{\delta \theta_{a}}{\delta t} = \frac{\partial}{\partial z} \left( K_{a} \frac{\partial \theta_{a}}{\partial z} \right)
\]

\[
\left( \theta_{w} - \theta_{w}^{j+1, m} \right) = \rho_{a} \left( \frac{1}{K_{a}^{j+1, m}} \frac{\delta K_{a}^{j+1, m+1}}{\delta z} \right) \frac{\delta \phi_{w}^{j+1, m+1}}{\delta t}
\]

\[
\left( \theta_{a} - \theta_{a}^{j+1, m} \right) = \rho_{a} \left( \frac{1}{K_{a}^{j+1, m}} \frac{\delta K_{a}^{j+1, m+1}}{\delta z} \right) \frac{\delta \theta_{a}^{j+1, m+1}}{\delta t}
\]

\[
\left( \theta_{w} - \theta_{w}^{j+1, m} \right) = \rho_{a} \left( \frac{1}{K_{a}^{j+1, m}} \frac{\delta K_{a}^{j+1, m+1}}{\delta z} \right) \frac{\delta \phi_{w}^{j+1, m+1}}{\delta t}
\]

\[
\left( \theta_{a} - \theta_{a}^{j+1, m} \right) = \rho_{a} \left( \frac{1}{K_{a}^{j+1, m}} \frac{\delta K_{a}^{j+1, m+1}}{\delta z} \right) \frac{\delta \theta_{a}^{j+1, m+1}}{\delta t}
\]
where $\Delta t = t^{j+1} - t^j$ is the time step, the superscript $m$ denotes the iteration level, and $\delta p_\alpha$ is the pressure head difference of phase $\alpha$ (water or air) in the iteration step:

$$\delta p_{\alpha}^{j+1,m+1} = b_{\alpha}^{j+1,m+1} - b_{\alpha}^{j+1,m}$$  \[A5\]

This parameter is also used for determining the convergence of the iteration and moving to the next time step.

Following the general procedure of Celia and Binning (1992), the Picard iteration method was used to linearize the equations of water capacity ($C_w$), hydraulic conductivity ($K_w$), and the soil’s air conductivity ($K_a$).

The capillary pressure head (Eq. [2]) was calculated at each iteration step and was subsequently used to determine the non-linear functions. For consistency with the assumptions invoked here, we used the soil’s air (Eq. [3]) and the hydraulic (Eq. [4]) conductivity functions presented above. For the same reason, the air conductivity was assumed to depend linearly on the air content:

$$\theta_a = \frac{K_a}{k_a} + \theta_a = \frac{K_a^0}{k_a} \exp(\alpha_a b_a)$$  \[A6\]

The minimal air content ($\theta_a$) value of $K_a^0/k_a$ obtained at $b_a = 0$ is a direct consequence of the minimal, finite air conductivity at water saturation (derived from Eq. [4]) and of the assumption of a linear relation between air conductivity and air content. This value may be regarded as part of an effective residual air content ($K_a^0/k_a + \theta_a^R$),

$$\theta_a = \theta_a - \theta_a^R$$  \[A7\]

Therefore, the specific water capacity is

$$C_w = \frac{d\theta_w}{db_a} = \frac{\alpha_w K_a^0}{k_a} \exp(\alpha_a b_a)$$  \[A8\]

A boundary condition of sinusoidal air flux was assigned at the bottom of the 1D soil domain ($Z = 0$) along with a constant, atmospheric air pressure head ($b_a = 0$) at the soil surface ($Z = L$). Additionally, a constant water pressure head was prescribed at both ends of the soil domain:

$$\frac{\partial b_a}{\partial z} = -\frac{q_0}{k_a} \sin(\omega_0 T), \text{ at } Z = 0$$  \[A9\]

$$b_a = 0, \text{ at } Z = L$$

$$b_a = \phi_w, \text{ at } Z = 0$$

$$b_a = \phi_w - l, \text{ at } Z = L$$

where $L = 2L/\alpha_a$ is the ND height of the soil domain.

The example described in Fig. 3 was obtained using the following parameters: total ND water head of $H_w = -0.25$, soil parameters of $\alpha_a = 0.02 \text{ cm}^{-1}$, $K_a^0 = 0.01 \text{ cm s}^{-1}$, and $k_a = 1 \text{ cm s}^{-1}$, and a mean air flux of $q_0 = 0.06 \text{ cm s}^{-1}$. In addition to these parameters used for the two-phase numerical model (and also in the analytical solutions), the following parameters were assigned: water content at saturation $\theta_w = 0.4$, $K_a^0/k_a + \theta_a = 0.05$, hydraulic conductivity at saturation $K_w^0 = 0.02 \text{ cm s}^{-1}$, and $\alpha_w = 0.01 \text{ cm}^{-1}$.

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**References**


