A Macroscopic Analytical Model for Pressure Wave Propagation in the Water of a Variably Saturated Porous Medium

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A macroscopic one-dimensional analytical model has been developed for the attenuation of low-frequency pressure waves in the water of a variably saturated porous medium. The developed model associates pressure amplitude attenuation with pore scales. Pressure propagation in a single pore size is described by a linear wave equation, and the composition of characteristic pore sizes are combined for representing the porous medium. The capillary bundle approach allows for a geometrical abstraction of the pore sizes and the definition of the water-filled capillaries under different water content states. This approach allows the pore water pressure attenuation to be addressed directly with readily available soil properties and parameters. Theoretical findings backed by experimental observations in three soils under saturated conditions indicate that the pressure wave attenuates more rapidly in the water of fine-textured soils than coarser ones. Furthermore, the results indicate that the effect of larger pores on pressure attenuation is significant at high water contents, but the pore-size distribution and the smaller pores dominantly control the pressure attenuation as the water content decreases. The developed model is consistent with previous studies and with the theory of Biot for waves in composite media.

Modeling the propagation of waves in porous media is a vast research field with potential ramifications in multiple related physical models and scales. For some applications, such as nondestructive medical testing, observing wave propagation through porous media can be harnessed to identify qualities and deformations of the porous material without destructive intrusion (e.g., Haire and Langton, 1999). In other cases, such as oil production, waves can potentially be used for oil recovery, encouraging a better understanding of their impact on the mobilization of fluids with different viscosities in porous media (e.g., Chrysikopoulos and Vogler, 2006).

Most reported studies in the field of wave propagation in porous media are based on either the fundamental studies of Biot (1956, 1962) or the premise of mixture theories. Biot’s biphasic theory was further developed and generalized with three-phase models (e.g., Lo et al., 2005). Many studies have addressed the conservation equations and the constitutive relations for each phase and their interactions, calibrating models of wave propagation with observations concerning the entire bulk. Additionally, some studies focused on specific phases comprising the porous bulk to gain an understanding of the effects of wave phenomena on state variables in these phases. Specifically, fluid-phase state variables such as pressure and velocity, and their effect on the motion of solutes and particles, were studied both theoretically (Sorek, 1996) and experimentally (e.g., Thomas and Chrysikopoulos, 2007; Gross et al., 2003). However, it is not trivial to use such existing theories and models for extracting the influence of a pressure wave in the bulk on state variables of a specific phase (e.g., Steeb et al., 2014), as this requires confronting a complete set of balance equations of all phases and therefore requires detailed knowledge of the porous medium and the constitutive relations among phases.

A pressure wave propagating in the fluid phase can be characterized as a Biot-type slow wave that is associated with fluid displacement relative to the solid skeleton. Low-frequency
acoustic waves in the fluid phase are highly attenuated across short distances and propagate at velocities that are usually associated with pressure diffusion (Zhang and Ping, 2018). Biot’s theory poorly predicts the high attenuation of these low-frequency waves (Goloshubin and Korneev, 2000). Addressing low-frequency pressure waves in saturated porous media as a mechanism similar to the diffusion of pressure from an oscillating pressure source provided the foundation for modeling this phenomenon (Norris and Rebinsky, 1993). The new approach highlighted the involvement of this mechanism in different geophysical observations (e.g., Rasouli and Sutherland, 2014; Sminchak and Gupta, 2003).

Because the wave velocity of low-frequency waves is relatively slow in soils, their wavelength can be on a scale of ~10 m, which is one to two orders of magnitude larger than the distance scale in which these waves attenuate almost completely (Yang et al., 2015). Consequently, it is challenging to find the proper spatial scale for experimentally monitoring these waves. Nevertheless, measuring and predicting the propagation of pressure perturbations across relatively small scales may be useful for applications where waves are utilized for inducing mass mobility such as liquids, solutes, or particles in porous media.

Our objective was to follow the spatial amplitude–attenuation of pressure waves in the water phase. For that purpose, we developed a one-dimensional macroscopic model and validated it with experiments. By neglecting the temporal variations in the pressure and focusing on its maximum only, the solution of the wave equation is simplified to an exponential-type attenuation equation. Considering pores as pipes, we use the capillary bundle approach to represent the pore-structure geometry and use soil mechanical properties for representing the wave characteristics. The soil hydraulic properties are incorporated for determining which capillaries are filled with water at different soil water contents. We present an analytical model using macroscopic equations. Its novelty is in the use of a simple wave equation that is known for pipes together with macroscopic averaging for directly extracting the pressure in the water phase.

### Theory

Consider a three-phase media, with water and air that occupy the pore space, and consider that the pressure wave in a single size pore can be explained with a macroscopic linear wave equation, such that the pressure propagation in the water phase of the bulk porous medium is the superposition of the pressure propagations in all pore spaces occupied with water composing a porous medium. The pressure attenuation in each pore space is assessed using a characteristic pore size. These characteristic pore sizes are extracted from the soil properties with the capillary bundle approach (Childs and Collis-George, 1950), which enables representation of the hydraulic and geometric characteristics of a porous medium. The capillary bundle approach is commonly used for understanding the relationship between unsaturated hydraulic properties and the geometry of porous media phases (Jury and Horton, 2004). Using the capillary bundle model, we gain a deterministic representation of a porous medium, in which different water contents are established by assessing which capillaries are water filled. Using this relationship between soil water content and water-filled capillaries, together with constitutive relationships between hydraulic and mechanical properties of the porous medium (water, air, and solid particles), we analyze the effect of water content on the attenuation of pressure by addressing pressure wave propagation through the water-filled capillary groups. This allows analysis of pressure attenuation with distance by using a distinctive absorption coefficient for each capillary group with its hydraulic radius. Consequently, the pressure at a given location consists of the pressure distribution across the water-filled capillary groups affected by mechanical properties of the bulk and the capillary group sizes.

Pressure waves propagate through all phases in the bulk porous medium, each phase with its unique wave characteristics. Considering variably saturated media, with solid particles, liquid water, and air phases, a pressure perturbation first gives rise to a fast-wave mode (P1) that exhibits in-phase motion of all bulk components. Following it, the pressure in the water phase propagates in a slower wave mode (P2), in which inertial effects create relative motion between the fluid and solid matrix, resulting in viscous dissipation of acoustic energy. In a capillary bundle conceptualization, the P2 wave mode propagates only through the water-filled capillaries (Fig. 1). This is the wave of interest in this study. In addition to it, a second slow-wave mode (P3) also exists, known as a capillary wave traveling on the water–air interface (Lo et al., 2005).

We calculate the attenuation of pressure for the water phase only, assuming energy loss in the interaction between the water and the solid (P2) and neglecting interaction between the water and the air (P3). These interactions and the energy dissipation caused by the air–water interface, which is linked to P3, are characterized by a traveling velocity significantly slower than that of the water–solid interface (P2). Hence, we assume that P3 does not affect the peak pressure in the water.

Whereas our model refers to the water phase, its predictions are achieved by accounting for mechanical, hydraulic, and geometrical properties of the porous medium as a bulk, characterized by the composition and distribution of all the phases.

The model consists of three principal stages: (i) the pore geometry and hydraulic properties of the porous medium are represented as a bundle of discrete capillary groups; (ii) the propagation of a pressure wave addresses each of the discrete capillary groups; and (iii) the pressure wave in the water phase is obtained by averaging of the amplitudes of the waves in the different capillary groups.

### Pressure Propagation in a Capillary

A pressure (acoustic) wave can be described by the linear damped wave equation (Landau and Lifshitz, 1987, p. 251–254)

\[
\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} + a \frac{\partial p}{\partial t}
\]  

[1]
where \( p \) \([\text{ML}^{-1} \text{T}^{-2}]\) is the pressure perturbation in the wave (considering constant background pressure), \( x \) \([\text{L}]\) is the spatial axial coordinate \( (x \geq 0) \), \( \alpha \) \([\text{LT}^{-2}]\) is a damping factor, \( t \) \([\text{T}]\) is time, and \( c \) \([\text{LT}^{-1}]\) is the wave velocity.

The pressure, \( p(x,t) \), of Eq. [1] can be replaced by the magnitude of the peak pressure as a function of space only, \( P(x) \). The common solution for the peak pressure is \( (\text{Cai et al., } 2018)\)

\[
P = P_0 \exp \left( -\gamma x \right) \quad [2]
\]

where \( \gamma \) \([\text{LT}^{-1}]\) is an absorption coefficient. The solution, Eq. [2], provides a description for the propagation of pressure originated by a peak pressure perturbation, \( P_0 \), at the source \( (x = 0) \), in a semi-infinite medium with a specific absorption coefficient. The absorption coefficient of acoustic waves is considered to be power-law frequency dependent, such that

\[
\gamma \propto \omega^\eta \quad [3]
\]

where \( \omega \) \([\text{LT}^{-1}]\) is the wave angular frequency, and \( \eta \) (dimensionless) is a real non-negative constant that is a function of the medium’s properties.

We consider Eq. [2] and seek the absorption coefficient for a pressure wave traveling in the fluid inside a solid capillary. We assume that the walls of the capillary absorb most of the wave energy and therefore cause the major attenuation of the wave. Furthermore, we consider that the wave is reflected from the solid wall, thereby assuming that: (i) the density of the wall is significantly higher than that of the fluid so that the pressure wave does not penetrate the wall; and (ii) the temperature of the wall is constant because the absorbed energy in the wall is relatively small considering its specific heat. In water, temperature fluctuations due to compression are negligible and heat conduction is considered relatively insignificant. Under these assumptions, the absorption coefficient in Eq. [2] can be approximated as \( (\text{Landau and Lifshitz, } 1987, \text{p. } 300-304)\)

\[
\gamma = \sqrt{\frac{\alpha}{2}} \frac{\nu}{cR} \quad [4]
\]

where \( R \) \([\text{L}]\) is the capillary radius, and \( \nu \) \([\text{LT}^{-1}]\) is the fluid (water) kinematic viscosity. The wave velocity, \( c \), defined in Eq. [1], denotes hereafter the specific case of the sound wave velocity in the water inside the capillary.

Substituting the expression in Eq. [4] into Eq. [3] leads to a value of \( \eta = 0.5 \). This is suitable for porous media, given that \( \eta \) usually ranges between 0 and 2 for multiphase materials \( (\text{Cai et al., } 2018)\).

Our use of \( \gamma \) and its relationship with \( \omega^{0.5} \) takes a similar form as in previous studies \( (\text{e.g., Silin et al., } 2003)\). Furthermore, Eq. [4] relates \( \gamma \) to the hydraulic radius and the wave velocity, which is also similar to previous studies that showed it to be a function of the permeability of the porous medium, \( k \) \([\text{L}^2]\), and the compressibility of the bulk. These properties are analogous, as the permeability is proportional to the square of a representative hydraulic radius \( (k \propto R^2) \) under the Hagen–Poiseuille equation and Darcy’s law \( (\text{Bear, } 1972, \text{p. } 162-166)\), and the wave velocity and bulk compressibility are both functions of the solid and fluid mechanical properties. However, previous studies used a different approach, conducting a mass balance for Darcy flow in porous media and relating the pressure to the water mass through the bulk compressibility \( (\text{Barenblatt et al., } 1990)\), which resulted in a diffusion-type differential equation for the pressure. Applying an oscillating boundary condition for this diffusion equation leads to the solution of exponential attenuation for the pressure amplitude, as in our case. The same approach was also used for pressure fluctuations of air in porous media \( (\text{Fukuda, } 1955)\) by relating air pressure to the air mass directly through the compressibility of the air (instead of the bulk), resulting in the same principle expression.

Although the principle expression is similar to our analysis, there are fundamental differences in the final solution.

In the development here, each \( \gamma \) represents only a discrete part of the porous medium, and the attenuation of the total water pressure is calculated with a superposition of these discrete parts. Thus, although Eq. [1] is microscopic and is used for modeling the wave in single capillaries, the porous medium is represented by the
composition of capillary bundles and addresses the pressure wave motion in a macroscopic frame of reference. Consequently, the final solution is an average of the exponential equations for the different capillaries, as detailed in the derivation below.

**Porous Medium as a Bundle of Capillaries**

Consider a porous medium constructed from a bundle of several capillary groups, where each group characterizes pores of equivalent size in the medium as continuous capillaries. Each group consists of a unique number of identical capillaries with a specific radius and length. Water content and soil water retention are thus associated with the capillary bundle geometry. Furthermore, water content is correlated to the soil mechanical properties. Hence, we correlate between the hydraulic properties of the modeled medium, the capillaries’ geometry, and the pressure wave propagation.

For a given water content, \( q \), the effective saturation, \( S_{ej} \) (dimensionless), is defined as

\[
S_{ej} = \frac{\theta_j - \theta_r}{\theta_s - \theta_r}
\]

where \( \theta_s \) and \( \theta_r \) denote the residual and saturated water contents, respectively. The residual water content, \( \theta_r \), is considered the limit of hydraulic continuity in the medium, below which pressure can no longer propagate continuously in the water phase.

Consider the water content of a porous medium as a sum of discrete (and constant) water content intervals, \( \Delta \theta \), such that the saturated water content is distributed across \( j \) equal segments, each representing a group of identical capillaries. At a specific water content state, there are \( j \) capillary groups that are filled with water (\( 0 \leq j \leq J \)), defining a given water content as

\[
\theta_j = \theta_r + j \Delta \theta
\]

with the specific case for saturated water content,

\[
\theta_s = \theta_r + J \Delta \theta
\]

Substituting Eq. [6] and [7] into Eq. [5], we obtain the definition for the effective saturation by the number of water-filled capillary groups at a certain water content state, \( j \), and the total number of capillary groups in the medium, \( J \), as the ratio

\[
S_{ej} = \frac{j}{J}
\]

The relationship between the water content and the matric pressure head, \( \Psi \) [L], is the water retention curve. The radius of each capillary within a specific group is attained by using the retention curve. Using this relationship, for each value of \( \theta(\Psi) \) that coincides with a distinct water content of the capillary groups, a capillary radius, \( R_e \), is evaluated with the Young–Laplace equation as

\[
R_e = \frac{-2 \sigma}{\rho g \Psi}
\]

where \( \sigma \) [M T\(^{-2}\)] is the water–air surface tension, \( \rho \) [M L\(^{-3}\)] is the density of water, and \( g \) [L T\(^{-2}\)] is the gravitational acceleration. The capillary radius \( R_e \) represents the largest capillary group storing water at the \( j \)th water content state, \( \theta_j \) (with a matric pressure head of \( \Psi_j \)), such that capillaries with radii equal to or smaller than \( R_e \) are water filled and those with radii larger than \( R_e \) are empty of water.

The total water content is comprised of equal parts of water content intervals (\( \Delta \theta \)), such that each capillary group contributes the same water volume in the capillary bundle. Therefore, as the radius of the capillary group increases, the number of capillaries per group decreases. We note that the contribution of \( \theta_s \) to wave propagation is neglected because the capillary groups in the capillary bundle model describe the water content intervals \( D_q \) between \( \theta_r \) and \( \theta_s \) only, Eq. [5–7]. Thus, we assume that pressure in the water phase is completely damped at residual water content \( (\theta_r) \), similar to dry conditions.

**Geometrical and Mechanical Properties of the Modeled Medium**

The water-filled capillary groups are assigned a single and specific tortuosity that characterizes the water content state of the medium. At higher water content states, the tortuosity is lower because there are more capillary groups that are filled with water and better water connectivity. The tortuosity factor, \( \tau \) (dimensionless), is considered to be the ratio between the direct distance in the porous medium, \( d \) [L], and the effective length of a capillary, \( x \), such that \( x > d \) and

\[
x = \frac{d}{\tau}
\]

Substituting Eq. [10] and [4] into Eq. [2], we obtain the peak pressure in a single capillary as

\[
P = P_0 \exp \left( -\frac{\Delta \nu \sqrt{2\tau R}}{d} \right)
\]

The velocity of the sound wave in the water that resides inside the capillary bundle medium, \( c \), is a function of both the hydraulic properties and the mechanical properties of the bulk, which are functions of the properties of the different phases (solid, water, and air) and the water content. The wave velocity in the water within a capillary can be expressed by an extended form of the Newton–Laplace equation (Halliwell, 1963):

\[
c = \sqrt{\frac{1}{\rho \left( 1/K + D/eE \right)}}
\]

where \( K \) [M L\(^{-1}\) T\(^{-2}\)] is the water bulk modulus, \( E \) [M L\(^{-1}\) T\(^{-2}\)] is Young’s modulus of the capillary wall, \( D \) [L] is the inner diameter of the capillary, and \( e \) [L] is the wall thickness of the capillary. In a porous medium composed of capillaries, the ratio of the capillary inner diameter to its wall thickness, \( D/e \), could be expressed with the porosity of the medium, \( \Phi \) [L\(^3\) L\(^{-3}\)] (which is effectively equal to the saturated water content, \( \Phi = \theta_s \)). Assuming that the
Consequently, the effective wave propagation in the water phase, with values of 
\[ \Phi = \frac{\pi (D/2)^2}{\pi (D/2 + e)^2} \] \[ 13 \]
such that
\[ \frac{D}{e} (\Phi) = \frac{2}{\sqrt{\Phi - 1}} \] \[ 14 \]

We examine this relationship by considering two limiting cases: (i) with values of \( E \) such that \( E/K > 1 \) (e.g., a steel pipe) and/or (ii) with \( D/e < 1/K \) (e.g., very small porosity, \( \Phi < 1 \)); in both of these cases, Eq. \[ 12 \] simplifies to the standard Newton–Laplace equation:
\[ \epsilon = \sqrt{\frac{K}{D}} \] \[ 15 \]

In reality, however, the value of \( E \) for soils is three orders of magnitude smaller than the value of \( K \) for water, such that \( E/K < 1 \). Consequently, the effective wave propagation in the water phase, Eq. \[ 12 \], is much slower than the wave in unconfined (free) water, Eq. \[ 12 \]. Thus, this set of hydraulic and mechanical parameters is assumed to be two orders of magnitude smaller than that of free water.

Pressure in the Pore Water: Averaging Capillary Groups

The source pressure in the water, \( P_0 \), is assumed to be distributed evenly to all water-filled capillaries. Subsequently, the pressure propagation is evaluated independently for each capillary size group, using Eq. \[ 11 \]. The pressure at distance \( d \) from the source is considered the average pressure of all water-filled \( j \) capillary groups at that location. Thus, for each water content, the relative pressure, \( P/P_0 \), is
\[ \frac{P}{P_0} = \frac{1}{\sum_{j=1}^{\text{cap}} \exp \left(-\frac{\sqrt{\Phi - 1}}{\sqrt{2F_0 R_j \tau_j}} \right)} \] \[ 16 \]

We assume that \( P/P_0 \) does not depend on the magnitude of \( P_0 \) such that the attenuation of the pressure perturbation, \( P \), is linear to the pressure perturbation at the source, \( P_0 \). Furthermore, the solution of Eq. \[ 16 \] is generalized for any background pressure for which the source is applied. We caution this generalization because it implies linearity for all cases. This can be violated in some cases—for example, if the background pore pressure is close to a point at which the intergranular stress may be affected (Bishop and Skinner, 1977). Under such conditions, the model assumptions would be valid for a pressure perturbation range that is smaller relative to a state in which the background pore pressure is lower.

Materials and Methods

Seven soil types were investigated in this study: four for theoretical analysis of the model (Table 1) and three for the experiments (Table 2).

The entire set of hydraulic and mechanical properties of the soils used here is taken from the literature and assumed as known properties. These are used as model inputs for simulating the wave propagation in these soils. Namely, the hydraulic properties include the retention curve parameters that produce the radii distribution for the capillary groups, Eq. \[ 9 \], and the mechanical properties are Young’s modulus of the soils, used for evaluating the wave velocity, Eq. \[ 12 \]. Thus, this set of hydraulic and mechanical parameters is assumed as the inherited properties of the medium. In contrast, the wave properties, the angular frequency \( (\omega) \) and the source pressure magnitude \( (P_0) \), are treated as unknowns. The wave frequency is considered constant for the theoretical analysis and is estimated via parameter optimization with experimental pressure observations (Appendix A). Whereas in the theoretical analysis we evaluate the relative pressure, \( P/P_0 \), directly, thereby eliminating the need for the magnitude of \( P_0 \) in the experiments the source pressure magnitude \( (P_0) \) is estimated via parameter optimization with the wave frequency (Appendix A).

For comparison between pressure propagation in soils with different properties, we make use of the wave penetration depth, \( d_{\text{pen}} \), which is defined as the location where the relative pressure magnitude decays to a value of \( 1/e \) \( (P/P_0 \approx 0.37) \).

Model Parameters

The model solution, Eq. \[ 16 \], was analyzed using the soil hydraulic and mechanical properties of two sands and two silts.

### Table 1. Soil hydraulic and mechanical properties from Lu and Kaya (2014).

<table>
<thead>
<tr>
<th>Soil</th>
<th>Hydraulic properties†</th>
<th>Mechanical properties‡</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha ) ( n )</td>
<td>( \theta_r ) ( \theta_s )</td>
</tr>
<tr>
<td></td>
<td>m(^{-1})</td>
<td>— [L(^3) L(^{-3})] —</td>
</tr>
<tr>
<td>Esperance sand</td>
<td>2.2 2.52</td>
<td>0.018 0.39</td>
</tr>
<tr>
<td>Orta sand</td>
<td>2.3 2.4</td>
<td>0.004 0.38</td>
</tr>
<tr>
<td>Bonny silt</td>
<td>0.6 1.5</td>
<td>0.02 0.47</td>
</tr>
<tr>
<td>Elliot Forest silt</td>
<td>0.1 1.64</td>
<td>0.01 0.39</td>
</tr>
</tbody>
</table>

† From the model of van Genuchten (1980): \( \alpha \) and \( n \), characteristic parameters; \( \theta_r \) and \( \theta_s \), residual and saturated volumetric water contents, respectively.

‡ From the model of Lu and Kaya (2014): \( E_r \) and \( E_s \), Young’s modulus under residual and saturated conditions, respectively; \( z \), soil-specific characteristic parameter.

### Table 2. Hydraulic properties of van Genuchten, including characteristic parameters \( \alpha \) and \( n \) and the residual and saturated volumetric water contents \( (\theta_r \) and \( \theta_s \), respectively) for Hamra, fine sand, and loess soils. For the mechanical properties under saturated conditions, the Young’s modulus of the Hamra soil and fine sand was taken as \( E_s = 3 \) MPa, which is typical for sandy soils, and \( E_s = 1 \) MPa for the loess, which is typical for loamy soils (Lu and Kaya, 2014).

<table>
<thead>
<tr>
<th>Soil</th>
<th>( \alpha )</th>
<th>( n )</th>
<th>( \theta_r )</th>
<th>( \theta_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamra†</td>
<td>9.9</td>
<td>1.8</td>
<td>0.03</td>
<td>0.42</td>
</tr>
<tr>
<td>Fine sand†</td>
<td>1.7</td>
<td>6.4</td>
<td>0.07</td>
<td>0.36</td>
</tr>
<tr>
<td>Loess†</td>
<td>2.6</td>
<td>1.3</td>
<td>0.07</td>
<td>0.39</td>
</tr>
</tbody>
</table>

† From Russo and Bouton (1992).

‡ From Nachshon (2016).
(Table 1). The sandy soils, Ottawa sand and Esperance sand, differ in their pore-size distribution and their elastic modulus. The pore-size distribution of Ottawa sand is more uniform than that of Esperance sand, and the bulk elastic modulus dependence on water content is also different between them. The silt soils, Bonny silt and Elliot Forest silt, feature higher air-entry values than the sandy soils, due to their finer texture, and are characterized by wider pore-size distributions. The significant distinction between these silts is in their bulk elastic modulus, which differs in its magnitude and its dependence on water content. The properties of these four soils (Table 1) were used as inputs to the model, Eq. [16], enabling the simulation of the damping of a pressure wave for the different soils, representing a limited variety of natural hydraulic and mechanical properties.

Hydraulic Properties

The water content of each soil was segmented into $J = 100$ capillary groups (Appendix B). The radius distribution of the capillary groups is defined for a porous medium through its water retention curve, as described in Eq. [9]. For that purpose, the van Genuchten (1980) model provides the relationship between the effective saturation level, $S_e$, Eq. [8], and the matric pressure head in the form of

$$S_e = \left(1 + \left|\alpha \Psi \right|^n \right)^{-m} \quad [17]$$

where $\alpha$ (L$^{-1}$), $n$ (dimensionless), and $m$ (dimensionless) are empirical characteristic parameters. The value of $\alpha$ is related to the reciprocal of the air-entry value, $n$ is an index for the pore-size distribution ($n > 1$, as the value of $n$ is larger the pore-size distribution is more uniform), and $m = 1 - 1/n$.

The tortuosity factor is a function of water content, $\tau(\theta)$, estimated here for each water content of the $j$ groups in the capillary bundle of Eq. [16] as (Moldrup et al., 1997)

$$\tau_j = 0.66 \left(\frac{\theta_j}{\theta_s}\right)^{1/3} \quad [18]$$

We note that this approximation is not soil specific.

Mechanical Properties

The wave velocity in the soil water, Eq. [12], is considered a function of the soil’s Young’s modulus, $E$, which depends on the soil’s granular properties and the water content and can be estimated as (Lu and Kaya, 2014)

$$E = E_r + (E_s - E_r)S_e^z \quad [19]$$

where $z$ (dimensionless) is a soil-specific characteristic parameter, and subscripts $r$ and $s$ denote Young’s modulus values at residual and saturated water contents ($\theta_r$ and $\theta_s$), respectively.

Wave Properties

The angular frequency, $\omega$, and the peak pressure at the source, $P_0$, are the only model parameters that are not considered as inherent properties of the porous medium but rather of the applied external constraints. In the simulations of the four soils (Table 1), we assigned an arbitrary radial wave frequency value of $\omega = 125$ rad s$^{-1}$, constant throughout the different soils and water contents. Later, when we evaluated the model with experimental data, $\omega$ was estimated via parameter optimization (Appendix A).

Experimental Setup

Pressure propagation and attenuation of the model were compared with wave-pressure measurements for three additional soils (Table 2): fine-textured sand, Hamra (Red Mediterranean sandy loam), and a loess soil (sandy clay loam). Although both the fine sand and the Hamra are sandy soils, they display significantly different hydraulic properties. Compared with the fine sand, the Hamra soil has a wider pore-size distribution and a lower air-entry value. These soils were used in the experiments under saturated conditions only; thus, we used typical values of Young’s modulus for each of the three soils (Table 2). The soil experiments were conducted in a vertical column under saturated conditions (Fig. 2). The pressure of the wave in the water phase was measured using pressure transducers (Honeywell NBP series with a gauge pressure range of 0–0.41 MPa), located at the tip of water-filled tubes, which were installed along the

![Fig. 2. The experimental setup, consisting of an air pressure tube (a), a high-pressure water source (b), solenoid valve (c), vertical soil column (d), pressure transducers (e), a tube for saturation and drainage (f), and the data-acquisition system (g).](image)
column and perpendicular to its wall, penetrating from the outside to the column center. Before the experiments, the column was dry-packed with soil and subsequently saturated with water from its bottom. After saturation, the water outlet from the bottom was raised to the top of the column, obtaining hydrostatic conditions. The top of the column was connected to a water line that enabled short-duration pressure applications with different pressure magnitudes through a solenoid valve (standard irrigation valve, S1603 Galcon). The other end of the solenoid was connected to a pressurized water container holding water at pressures of up to 0.15 MPa gauge above atmospheric pressure. A data-acquisition system (CR10, Campbell Scientific) controlled the valve and monitored the pressure. During the experiment, pressure pulses were created in the pore water by opening and closing the solenoid valve at the top of the column, resulting in abrupt changes of pressure. Different pulse durations were applied by altering the opening of the valve, during which the propagation of the pressure perturbation along the column was monitored. The pressure perturbation generated flow in the column, and water drainage was collected from its bottom. This setup, with water entering the column top with abrupt pressure pulses, allowed constant water content under saturated conditions only.

Wave Characteristics

As mentioned above, the wave characteristics are not inherent properties of the porous medium but rather of the applied external constraints. In the experiments, it was practically impossible to measure the pressure rise at the exact point of entry to the soil. Furthermore, estimating the frequency was also challenging due to the reaction between the pressure pulse and the porous medium. Therefore, for simulating the experimental conditions, the pressure attenuation of the model was fitted to the measured data points by optimizing the angular frequency and the peak pressure at the entry to the soil (Appendix A).

Results and Discussion

Model Predictions of Pressure Attenuation

We analyzed the effects of different soil hydraulic and mechanical characteristic properties (Table 1) on the simulated attenuation of a pressure wave in the soil water as a function of distance from the pressure source and water content.

The relative peak pressure amplitude of the pressure wave, \( P/P_0 \), in Ottawa and Esperance sand is presented for different discrete capillary size groups as a function of distance \( d \) from the pressure source (Fig. 3). The different capillary sizes represent different segments that comprise the total water content of the soil, as in Eq. [9] and [17]. In this specific example, the soils are under saturated conditions, where all capillary groups are filled with water, and the pressure attenuation in Fig. 3 is shown for specific capillary groups, which were chosen for presentation purposes.

Figure 3 shows that the pressure attenuation per distance is more significant for the smaller size capillaries. This is because the absorption coefficient is relatively large for small capillaries due to the assumption of Eq. [4] that the walls of the capillary absorb most of the wave energy. Furthermore, the soil type influences the pressure attenuation. In Ottawa sand, where the pore-size distribution is fairly uniform, the capillary sizes are more similar to each other compared with those of Esperance sand. As a result, the change in attenuation between the capillary groups in Ottawa sand is less noticeable than that of Esperance sand.

The pressure attenuation as a function of both water content and distance from the source is presented for Ottawa sand (Fig. 4). For each combination of water content and distance, the pressure attenuation was evaluated by volume averaging with Eq. [16]:

(i) calculating the pressure attenuation per capillary group in the...
water-filled capillary groups at that water content; and (ii) averaging these pressures of the water-filled groups.

The result for saturated conditions ($Se = 1$) in Ottawa sand (Fig. 4) represents the average of pressures from all the capillary groups at saturated water content (Fig. 3, solid lines). As the water content decreases, it is apparent that pressure attenuation with distance away from the pressure source is more noticeable. For low water contents, the smaller pores are a larger fraction of the water-filled capillaries, causing the attenuation to be more significant. In addition to the direct effect of the capillary size, the tortuosity also impacts the attenuation. As the water content decreases, the tortuosity increases and the capillary length becomes longer per bulk length of porous medium, so the pressure attenuates at shorter distances in the bulk soil.

Impact of Hydraulic Properties on Pressure Attenuation

We analyzed the effect of soil texture on pressure attenuation. Soil texture and its geometry are reflected in the pore-size distribution and air-entry value properties that manifest their effect on the water distribution. The combined effect of the soil texture and hydraulic properties can be evaluated by observing the simulated pressure attenuation for different soil types (Fig. 5). For fine-textured soils that are characterized by relatively small pore sizes, the pressure attenuates more noticeably than in the coarse-textured soils. Furthermore, the uniform grain and pore sizes of the sandy soils display a smooth, exponent-like attenuation curve of the pressure. In contrast, the silts, which are characterized by a wide distribution of pore sizes, display a noticeably different-shaped pressure attenuation curve. The attenuation in these soils is characterized by an initial sharp drop in the pressure, followed by a flatter pressure-attenuation curve, where the pressure drop is less pronounced per distance. The initial drop in pressure at short distances is attributed to the smaller pores (capillaries) of the soil. Whereas some portion of the pressure is initially damped, as the distance from the pressure source increases, the pressure damping is less pronounced (per distance) because attenuation is generally governed by the larger pores. As the pore-size distribution is wider, there are larger size differences between the small and large pore sizes in the soil (as in the silts compared with the sands). Consequently, in soils with relatively wide pore-size distributions, the shape of the pressure-attenuation curve is noticeably different than a smooth exponential-type curve, with a sharp drop at short distances and a long flat tail at longer distances. Another notable difference among the soils is the impact of water content on attenuation. Generally, all the soils attenuate the pressure more significantly when the water content decreases initially from saturation. This is evident in the results, comparing the attenuation per distance between saturated conditions ($Se = 1$, Fig. 5a) and near-saturated ($Se = 0.9$, Fig. 5b). However, after the initial pressure drop near the source, at a distance of a few centimeters, the attenuation is smaller in the sands than the silts.

The variations in hydraulic properties are relatively minor between the two sands compared with the noticeable differences between the sands and the silts. Yet, slight differences in the pressure attenuation are noticeable between those sands (Fig. 5). Close to the source, the pressure attenuation in Esperance sand is slightly more significant than in Ottawa sand. However, after a certain distance, the pressure attenuates more significantly in the Ottawa sand than in the Esperance sand. The distance of the intersection between the pressure attenuation curves of these sands increases at lower water contents. At any given location (distance from the source), the pressure attenuation in Esperance sand is affected more noticeably by changes in water content. We attribute this to the pore size magnitude and distribution. An inherited feature of the model is that all the capillary groups have the same cumulative surface area. Therefore, in cases where the pore-size distribution features small radius changes among the

![Fig. 5. The relative pressure, $P/P_0$, with distance for Ottawa sand, Esperance sand, Bonny silt, and Elliot Forest silt evaluated at effective saturation levels of (a) $Se = 1$, (b) $Se = 0.9$, and (c) $Se = 0.8$.](image-url)
capillary groups, as in the case with Ottawa sand, the averaging of the pressure propagation in the groups at different water contents results in small changes in the total water pressure. In general, as the variability in pore sizes increases and the pore-size distribution is less uniform, the difference in the pressure propagation between water contents increases (at a specific distance). Still, it would be wrong to assume a monotonic trend of pressure attenuation as a function of pore-size distribution only. Because the attenuation of pressure and its dependence on water content is based on the nonlinear expressions of Eq. [16] and [17], the propagation of pressure is complex. Yet, the property that has the main impact on the relationship between the change in attenuation and water content is the pore-size distribution, denoted by the shape parameter \( n \) in Eq. [17]. The pore size distribution is relatively narrow in Ottawa sand compared with Esperance sand (Table 1). Consequently, in Ottawa sand, the attenuation is relatively similar between the pore sizes (capillary groups), and therefore the effect of water content on pressure attenuation is less noticeable (Fig. 5). This phenomenon is further demonstrated by analyzing the sensitivity of pressure attenuation in Ottawa sand to the parameter \( n \) (Fig. 6). The pressure attenuation for three \( n \) values is reported for three water contents (Fig. 6a–c). For a relatively uniform pore-size distribution, with \( n = 4 \) (solid line in Fig. 6), the change in attenuation between water contents is moderate. For a less uniform pore-size distribution \( (n < 4) \), the attenuation is more noticeable as the water content decreases, entailing small capillaries with higher damping of the pressure with distance. Already at full saturation (Fig. 6a), the pressure attenuation in the wider pore-size distributions \( (n = 2 \text{ and } 1.5) \) is noticeably different from the one in the more uniform pore-size distribution \( (n = 4) \). The pressure for \( n = 2 \text{ and } 1.5 \) attenuates more drastically initially, at the first few centimeters, but is less attenuated at farther distances, approximately 30 cm from the pressure source. This is an outcome of the wider pore-size distribution that introduces both smaller and larger pores into the medium, especially under saturated conditions where all the capillary groups are water filled. Because smaller capillaries have greater absorption coefficients, the pressure is significantly attenuated at short distances by the smaller pores. In contrast, the larger pores give way for pressure to propagate to a farther distance because the pressure is less attenuated in these pores. In cases where all the pores are of similar size, the attenuation of the relative pressure, \( P/P_0 \), can be rationalized as having a single absorption coefficient for the entire medium. As the pore-size distribution becomes wider, the attenuation varies, featuring an assortment of absorption coefficients per attenuation.

The parameter \( \alpha \) in Eq. [17] characterizes the air-entry value of the soil, which is also related to the pore sizes and specifically to the magnitude of the largest pore that characterizes the soil. The influence of the air-entry value on the pressure attenuation is demonstrated for three \( \alpha \) values at three water contents (Fig. 6d–f). As the value of \( \alpha \) increases (and the air-entry value decreases), the soil has larger pores in its pore-size distribution and attenuation becomes less noticeable (Fig. 6). This trend is evident under both saturated and partly saturated conditions. However, the proportion of attenuation between water contents is practically independent of \( \alpha \). The air entry value, depicted by \( \alpha \), mostly characterizes the size of the large pores and less the distribution of pores. As already presented, the pressure attenuation at lower water

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![Fig. 6. The relative pressure, \( P/P_0 \), with distance at three effective saturation levels, \( Se = 1, 0.9, \) and 0.8, for three values of the characteristic parameter \( n = 1.5, 2, \text{ and } 4 \), and three values of the characteristic parameter \( \alpha = 1.5, 1.9, \text{ and } 2.3 \text{ m}^{-1} \). Properties of Ottawa sand were used with fixed parameters apart from (a–c) \( n \) and (d–f) \( \alpha \). The solid line represents the same hydraulic properties \( (n = 4, \alpha = 2.3 \text{ m}^{-1}) \) in all plots.](image-url)
contents is more affected by the pore-size distribution, depicted by the parameter $n$.

Ottawa sand was arbitrarily selected for this analysis (Fig. 6), providing a set of constant parameters based on real values along with the varying parameters that are used for the sensitivity analysis. Conducting such a sensitivity analysis using constant parameters from other soil types results in similar trends.

Following the analysis conducted for understanding the impact of the pore-size distribution and the air-entry value separately, we assess the influence of the hydraulic properties in their various combinations on the attenuation distance. For that purpose, we use the definition of wave penetration depth, $d_{\text{pen}}$. Figure 7 depicts $d_{\text{pen}}$ for a range of $n$ and $\alpha$ values and their combinations. For a specific pore-size distribution (constant $n$ value), the decrease in penetration depth displays a quasi-linear relationship with the decrease in $\alpha$. Therefore, it may be generalized that for porous media with similar pore-size distributions, the functional relationship between the penetration depth of a pressure perturbation and the air-entry value is inversely linear (i.e., $d_{\text{pen}} \propto \alpha$). Thus, a linear relationship exists between the penetration depth and the largest capillary radius (Young–Laplace, Eq. [9]), which is equivalent to the linear proportion $d_{\text{pen}} \propto R$ (Fig. 7).

It is important to note that for representing the geometry of fine-textured soils the capillary bundle model has some significant physical limitations. These soils are usually associated with a wide pore-size distribution (low $n$ values) and a high air-entry value (low $\alpha$ values). The capillary bundle model is limited for depicting narrow and highly tortuous paths concerning pore connections, which often characterize fine-textured soils. The model assumptions concerning water phase continuity and matching effective hydraulic radii for the pore spaces are more suitable for relatively simple geometries, more common in coarse-textured soils, and less valid for fine-textured soils (Hoffmann-Riem et al., 1999). Furthermore, the structural complexity of unsaturated porous media is somewhat accounted for in the model by introducing a tortuosity factor, but the postulate of hydraulic continuity becomes less valid for depicting soils at low water contents.

Impact of Mechanical Properties

The pressure attenuation is influenced by the mechanical properties of soils, which are a function of the solid phase and the water content. Young's modulus depends on water content, e.g., Eq. [17], and determines the effective wave velocity in the fluid phase, Eq. [11]. The water content, represented here by the effective saturation ($S_e$), impacts Young’s modulus ($E$) and the wave velocity, $c$ (Fig. 8). For a given soil at prescribed effective saturation, a single constant wave velocity is considered for all the water-filled capillary groups. The power-law relationship between Young's modulus $E$ and effective saturation $S_e$ can be approximated as $E \propto S_e^{-x}$, where $x$ is a constant that depends on the soil type and the water content. Similarly, the wave velocity $c$ is related to the effective saturation $S_e$ through a power-law function $c \propto S_e^{-y}$, with $y$ being another constant.

Fig. 7. The penetration depth for different values of $\alpha$ and $n$ under saturated conditions. Properties of Ottawa sand were used with fixed parameters, apart from $n$ and $\alpha$, which vary in the range of values given in Table 1.

Fig. 8. The (a) Young’s modulus and (b) wave velocity in the soil water as a function of effective saturation for Ottawa sand, Esperance sand, Bonny silt, and Elliot Forest silt.
modulus and the effective saturation, Eq. [17], is depicted in Fig. 8a. Thus, a power-law relationship is observed also between the wave velocity in the water and Se (Fig. 8b). This relationship is similar to other studies that found a constitutive power-law relationship between the wave velocity in the bulk and the water potential (Lu and Sabatier, 2009). The elasticity of sandy soils, in particular, is almost independent of water content, and consequently, the wave velocity remains nearly constant for different saturation levels (Fig. 8b). Young’s modulus of the silt soils is somewhat different, as it decreases with increasing saturation levels (Fig. 8a), resulting in a corresponding decrease in the wave velocity (Fig. 8b).

The magnitudes of the calculated velocities (Fig. 8b) are on the order that characterizes a Biot-type slow wave (P2) at low-range frequencies (Steep et al., 2014; Lo et al., 2005; Carcione et al., 2004) and are in accordance with the general theory of waves in porous media (Yang et al., 2015).

Impact of the Wave Angular Frequency on Model Predictions

The wave angular frequency, \( \omega \), is treated such that a single and constant frequency is prescribed for a given soil, for all the capillary groups, irrespective of water content. Thus, the angular frequency, \( \omega \), remains constant across the water content range. This is in contrast to the wave velocity, \( c \), which depends on the water content. The influence of the wave frequency on pressure propagation, given in Eq. [3] and [4], entails that the absorption for all the capillary groups has the same power-law functional relationship with the angular frequency. Therefore, as commonly assumed, the pressure attenuation per distance is more pronounced for higher angular frequencies of the wave (Yang et al., 2015; Cai et al., 2018; Lo et al., 2007).

Pressure Measurements and Model Evaluation

We evaluate the model by comparing its predictions with experimental measurements of pressure attenuation in three different soils under saturated conditions. In the experiments, temporal pressure measurements were conducted at specific locations along the soil column (Fig. 9). At each location, there were noticeable differences in the duration of the measured pressure rise from the background pressure to its maximum (peak pressure). That duration becomes longer as the distance from the source is farther (Fig. 9). However, the duration value does not indicate a change in the effective frequency at different locations along the column. Considering homogeneous soil conditions, with constant hydraulic and mechanical properties, we relate these observations to the progressive lag of the pressure rise that increases with frequency (Carslaw and Jaeger, 1959, Sect. 2.6).

Simulated model pressure attenuation curves were fitted to observations in fine sand (Fig. 10 and 11), Hamra, and loess (Fig. 11) soils. For each location of the observations, a normalized pressure was evaluated by using the ratio of the observed pressure that was measured directly and the source pressure, \( P_0 = P(d \approx 0) \). In the experiments, the pressure source, \( P_0 \), was generated with different pulse durations and pressure magnitudes. Nevertheless, regardless of the magnitude of \( P_0 \), the relative pressure attenuation with distance remained practically the same (Fig. 10, different symbols). This observation, that the relative pressure attenuation does not depend on the magnitude of the induced source pressure, is an important aspect for the validity of the model. It is directly related to the model assumptions that the pressure perturbation is a linear function of the pressure perturbation at the source, as in Eq. [16]. For applied pulse durations of 0.031, 0.094, and 0.156 s (Fig. 10a–c), the model matched frequencies (\( f = \omega / 2\pi \)) in the range of 5 to 60 Hz for the three soils. Longer pulse durations gave rise to lower frequencies and resulted in lower pressure attenuation with distance. Relating these results to the context of the Biot theory, the obtained frequencies distinctly belong in the low-range wave type. The frequency is classified as “low” by comparing it to the characteristic frequency of a porous medium, which is proportionate to the ratio between the kinematic viscosity of the fluid (water) and the permeability of the bulk medium. Even for soils with very high permeability, the limit for the characteristic low-frequency range is approximated in the order of several kilohertz (Zhang and Ping, 2018). The frequencies in this study are well below that limit.

For comparison among the three soils, the pressure propagation along the column is presented for these soils under the same pulse duration (Fig. 11). Observing the differences between the fine sand and the Hamra soil, a relatively sharp attenuation in pressure is evident in the Hamra soil at short distances (between the source and about 5 cm). This can be attributed to the smaller pores of the Hamra soil, which is characterized by a relatively wider pore size distribution compared with that of the sand. At longer distances, the pressure decline in the Hamra soil is not as steep as in the sand because the pressure at those distances is mostly attenuated in the larger pores of the Hamra soil, which are larger than those of the sand. These results are consistent with the sensitivity
analysis conducted on the pore-size distribution with the parameter $n$ (Fig. 6).

In the fine-textured loess, the major attenuation occurs at a significantly shorter distance than in the other soils. After the initial pressure drop, in the range of 2 to 5.3 cm from the top of the column, there is significantly less pressure attenuation. These results are consistent with the attenuation trend shown for the silts (Fig. 5) and are suitable for the wide pore-size distribution of this loess ($n = 1.3$). Measurements were not possible at distances shorter than 2 cm due to technical issues of proximity to the column top or at distances farther than ~5 cm due to the low pressure magnitudes, which were below the sensitivity range of our sensors. In the work of Fukuda (1955), similar pressure measurements of propagating air-pressure pulses in sand were successfully modeled with a single exponential term relating to the estimated permeability of the medium. However, it is apparent from our results that the shape of the water pressure attenuation with distance cannot be accurately described with a single exponential function. Specifically, the influence of variations in porous media geometry on the water pressure propagation, which are reflected in our model (Fig. 11), will have no representation when relying only on the effective permeability of the bulk. Despite the differences in the analytical expressions between these studies, it is interesting to compare the penetration depth of the air-pressure measurements with the penetration depth of pressure that we observed in the water (~50 cm for air and ~10 cm for water). These differences can be partially explained by the ratio $\sqrt{(\omega \beta / \epsilon)}$ in the absorption coefficient, which incorporates the differences in the fluids and the differences in the pulse durations. This ratio is four times larger for water in our measurements than for air in a similar scenario (Fukuda, 1955). Therefore, for a given single capillary radius or an effective permeability, the exponential attenuation of the relative pressure will be at a distance four times shorter for water than for air, which is similar to the differences between the scales of penetration depth.

**Conclusions**

A macroscopic model was developed for simulating the pressure wave evolution in pore water, describing its peak pressure propagation and attenuation as a result of a pressure perturbation in the soil. The model uses the capillary bundle approach for mimicking soil structure and water retention; it was implemented to simulate pressure attenuation in four soils, and its predictions were
tested against pressure observations in three additional soils under saturated conditions.

Our main findings include the following:

- The peak pressure in the wave attenuates exponentially as a function of distance from the source in soils with a uniform pore-size distribution (i.e., similar pore sizes). When the pore-size distribution is less uniform, there is a distinct difference between the pronounced attenuation at a short distance, due to smaller pores, and a moderate attenuation at a longer distance in the larger pores.

- The change in pressure attenuation is correlated to the pore-size magnitude and distribution, together with the water content state. For example: (i) in soils with a wide pore-size distribution, a decrease in water content leads to a significant reduction in the average characteristic size of the pores containing water and therefore results in a substantial increase in the pressure attenuation; and (ii) the air-entry value of the soil, which addresses the size of the largest pores, provides an assessment for the pressure attenuation under saturated or near-saturated conditions. However, for lower water contents, in which these larger pores are empty, the pore-size distribution is more important for determining the extent of pressure attenuation.

- The model allows an understanding of complex interactions, such as the combined effects of pore sizes and water content. For example, although in fine-textured soils the wave velocity increases at lower water contents (due to higher elastic modulus) implying less attenuation, the results indicate that the impact of the decreasing water content on the decrease of the characteristic pore apertures is more pronounced, showing significant attenuation at lower water contents.

Compared with previous studies that aimed to model wave propagation in porous media, the presented analytical model enables the pore-water pressure to be approached directly, requiring a relatively small set of parameters that are usually readily available and enables an assessment of pressure attenuation in the fluid phase of porous media.

The key element for the model development and its simplification is the discretization of the porous medium to multiple characteristic pore apertures. Utilizing the capillary bundle approach for this methodology, the problem in hand was significantly reduced, as it (i) is restricted to a one-dimensional semi-infinite domain, (ii) does not account for temporal variation, (iii) leans on the macroscopic approach with constitutive mechanical relations, (iv) neglects fluid displacement and skeleton deformation, and (v) assumes phase continuity. These assumptions naturally limit the variety of porous media to which the model is relevant. However, it allows relatively simple and straightforward use with porous media that fall within these assumptions. For addressing pressure propagation in fine-textured soils, which are commonly not well depicted with the capillary bundle approach, a different geometric perception should be used, such as assigning effective properties to the bulk medium (e.g., permeability), thus approaching the pore pressure propagation as a diffusion wave (e.g., Yang et al., 2015). This is part of ongoing research.

As evident from our measurements and simulations, the spatial scale for the propagation of these highly attenuated waves traveling in the pore water is on the order of ~0.1 to 1 m. The predicted values of the wave velocity in the model are suitable to the velocity magnitudes of pressure diffusion as in a Biot-type slow wave in the low-frequency range. This is consistent with previous studies and with the model assumptions that pressure propagation in the water is much slower and independent from the solid phase and that its angular frequency is in the low-frequency range.

We have demonstrated and verified the predictive capabilities of the model with measurements in three different soil types. Thus, addressing attenuation of low-frequency waves in the fluid phase, for which predictions by more comprehensive models often do not comply with observations, the proposed model produces reliable predictions and maintains correlations with Biot’s theory and mixture theory of porous media.

Appendix A

Parameter Optimization of Frequency and Source Pressure

The model was fitted to experimental data of the peak pressures along the column, \( P \), optimizing the source peak pressure, \( P_0 \), and the angular frequency, \( \omega \). Rearranging Eq. [16], and explicitly denoting the modeled pressure as \( P_{mod} \), yields

\[
P_{mod} = P_0 \sum_{j=1}^{N} \exp \left( -\frac{\omega \sqrt{V}}{2} \frac{d}{\tau_j} \right)
\]

We define the normalized least-squares objective function (OF) for minimization as

\[
OF = \sum_{i=1}^{N} \left[ \frac{P_{mod} (d_i, V) - P_{exp} (d_i)}{P_{exp} (d_i)} \right]^2
\]

where \( P_{exp} \) denotes the experimental pressure data points. The pressure of both modeled and experimental data are functions of distance from the pressure source, \( d \). In addition, the modeled pressure is also a function of the model parameters, defined with the vector \( \mathbf{V} \), as

\[
\mathbf{V} = (P_0, \omega, J, \theta, \theta_1, n, \alpha, E, E_1, z, \beta, \mathbf{Se})
\]

In Eq. [A2], each \( i \)th data point in the dataset of \( N \) observation points could be assigned different weights, but we did not bias the optimization and chose equal weights for the entire dataset. However, to balance the different magnitudes of the data, we used in Eq. [A2] a normalized form for the OF, normalizing the data with \( P_{exp} \). Apart from the pressure at the source, \( P_0 \), and the angular wave frequency, \( \omega \), the model parameters in Eq. [A3] are obtained directly from the hydraulic and mechanical properties of the soils (Tables 1 and 2). The parameters \( P_0 \) and \( \omega \) are optimized for minimizing the OF. During optimization, the value of \( P_0 \) is constrained by the measured pressure between the maximum...
pressure input in the water phase outside the column and the peak pressure in the column at the location closest to the source.

The computation and evaluation of the model were conducted with Python (2.7) software, using the Numpy (1.9.2) and Scipy (0.15.1) modules. The nonlinear least-squares parameter optimization, for minimizing Eq. [A2], was conducted using the built-in function optimize.minimize within the Scipy module.

Appendix B
Discretization of the Capillary Groups

The developed model is composed by assembling a finite number of capillary groups for describing a continuous change in water content. The chosen size of the water content interval determines the number of capillary groups and consequently the capillary sizes (radii), Eq. [6–9]. The discretization of the bundle to $J$ groups, and thereby their equivalent capillary radii, $R$, may influence the evaluation of the pressure propagation in the porous medium. Setting the water content intervals small, with a larger number of discrete capillary groups, is advantageous for representing the continuous variation of water content in soils. To assess the degree of discrete subdivisions on model predictions, the model-evaluated pressure was fitted against measured pressure using different assembling scenarios of discretization. A unique frequency was obtained for each of these discretization scenarios. The accuracy of the fit of modeled to measured pressure attenuation was almost independent of the number of capillary groups. However, the modeled frequency decreased as the number of those groups increased (i.e., a lower frequency for a more continuous representation with small water content intervals). Furthermore, the change in the predicted frequency is significant only when the number of capillary groups is relatively small. For example, changing the number of groups from 10 to 20 resulted in a 13% frequency decrease, but a further increase from 30 to 40 groups resulted in <5% frequency change only. Finally, when approaching 100 groups, the frequency changes were <1%. This trend was consistent across several simulations with different soil properties (data not shown), indicating that beyond ~30 to 40 assembling intervals, model predictions are similar. To avoid the influence of discretization on frequency in this study, and to achieve almost unvaried frequency, we used 100 intervals for discretizing the water content range.

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